Homework H13 Solution

1. In lecture L11, Gruebele showed that any time-dependent wavefunction $\Psi(x,t)$ can be written as a sum over stationary states (eigenstate times $e^{-\frac{i}{\hbar}E_n t}$ factor):

$$\psi(x,t) = \sum_n c_n \psi_n(x)\ e^{-\frac{i}{\hbar}E_n t} \qquad (1)$$

still satisfies the time-dependent Schrödinger equation. He claimed that the coefficients c_n can be calculated as $c_n = \int_{-\infty}^{\infty} dx\, \psi_n^*(x)\psi(x,t=0)$. Prove this by

a. Set t=0 in equation (1) (what happens to the exponential?) to get a simple formula for $\psi(x,t=0)$, the wavefunction at time 0.

b. Multiply both sides of the equation in (a) by $\psi_n^*(x)$, and then on both sides integrate over dx from $\pm\infty$ (you do not need to evaluate the integral).

c. Here's an interesting couple of facts about eigenfunctions of any differential equation of the type $\hat{A}\varphi_n = a_n\varphi_n$, which was proved in Math 285:

1) All eigenfunctions are orthogonal or $\int_{-\infty}^{\infty} dx\, \varphi_n^*(x)\varphi_m(x) = 0 \text{ } if\ n \neq m$.

2) Eigenfunctions are normalizable or $\int_{-\infty}^{\infty} dx\, \varphi_n^*(x)\varphi_m(x) = 1 \text{ } if\ n = m$.

(Remember? Suuuuure…there is a handout in the lecture L11 reading material online if you want to refresh your memory.)

Apply these two facts to your equation in (b) and prove the formula for c_n.

Solution:

a) Setting t = 0 and plugging into eq. (1) we have

$$\psi(x,t=0) = \sum_n c_n\psi_n(x)\ e^{(0)} = \sum_n c_n\psi_n(x) \qquad (1a)$$

b) Multiplying by $\psi_m^*(x)$ and integrating gives

$$\int_{-\infty}^{\infty} \psi_m^*(x)\psi(x,t=0)\,dx = \int_{-\infty}^{\infty} \psi_m^*(x)\sum c_n\psi_n(x)\,dx = \sum c_n \int_{-\infty}^{\infty} \psi_m^*(x)\,\psi_n(x)dx$$

$$= c_m \quad , \qquad (1b)$$

which is the desired result. (It doesn't matter whether our final result contains m's or n's or any symbol known to humanity as an index, it is simply a place holder for any integer quantum number you choose.) We have used the linearity of integration (to pull the sum out of the integral) and the orthogonality of eigenfunctions as stated in part (c) (only one term in the sum survives, when n=m) to arrive at the answer.

Turn in 2. Prove that the wavefunctions

$$\psi_M(\varphi) = \frac{1}{\sqrt{2\pi}} e^{iM\varphi} \qquad M = 0, \pm 1, \pm 2, \ldots \tag{2}$$

are orthogonal. **HINT:** Look at the orthogonality condition given in question 1 part (c). Do the integral and clearly show what happens if $N \neq M$.

Solution: We need to show that

$$\int_0^{2\pi} d\varphi\, \psi_N^*(\varphi)\, \psi_M(\varphi) d = 0 \tag{2a}$$

Note the integral here is not over x from -∞ to ∞, but over angle from 0 to 2π. Substituting the form of the wavefunction into equation (2a) gives

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi e^{-iN\varphi}\, e^{iM\varphi} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi e^{i(M-N)\varphi} = \frac{1}{2\pi i(M-N)} e^{i(M-N)\varphi}\Big|_0^{2\pi}$$

$$= \frac{1}{2\pi i(M-N)}\left[e^{i2\pi(M-N)} - 1\right] = \frac{1}{2\pi i(M-N)}[1-1] = 0 \tag{2b}$$

If the argument of a complex exponential is any integer multiple of 2π the result will be equal to 1. Prove this to yourself if it is not obvious (use Euler's Formula). Note there is one exception to the above: if M=N, then $e^{-iM\varphi}e^{iM\varphi}$=1, the integral equals 2π, canceling the 1/(2π) out in front – so the functions are not just orthogonal, but already normalized.

3. Angular momentum is a *vector* quantity. This statement is in the book, and you heard it in lecture. Lets briefly explore some mathematics associated with that statement. The general definition of angular momentum is

$$\vec{L} = \vec{r} \times \vec{p} \quad , \tag{3}$$

the cross product of the radial vector with the momentum vector. For example, for a *classical* electron circulating around a proton (a hydrogen atom), r would be distance between the proton and the electron, and p would be the momentum as the electron circles around the proton.

Using eq. (3), prove that the magnitude of L is mvr for a particle constrained to rotate in a circle in the x-y plane. To do this, write the components of $\vec{r}$ and $\vec{p}$ in vector form (ex: $\vec{r} = x\hat{\imath} + y\hat{\jmath}$), convert to polar coordinates (e.g. x = r cos), take the cross product and simplify.

Note: There is also an easier (but much less fun) way to prove this using a separate definition of the cross product in terms of the magnitudes of the vectors and a trig function. Going either route is an acceptable proof, but it behooves you to know both.

Solution: The cross product of two vectors can be obtained by writing the vectors as rows in a determinant (this should be familiar from your math courses):

$$\vec{r} \times \vec{p} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} \\ x & y \\ p_x & p_y \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} \\ r\cos\varphi & r\sin\varphi \\ -p\sin\varphi & p\cos\varphi \end{vmatrix}$$

$$= \{rp\cos^2\varphi + rp\sin^2\varphi\}\,\hat{k} = \{rp(\cos^2\varphi + \sin^2\varphi)\}\,\hat{k} = rp\,\hat{k} = mvr\,\hat{k} \quad (3a)$$

The magnitude of a vector is the square root of the sum of the squares of its components (that's a mouthful) but since the angular momentum here only has a single component that lies on the z-axis, the magnitude *is* the length of this component:

$$|L| = \sqrt{(mvr)^2} = mvr \qquad (3b)$$

An alternate definition of the cross product is given below:

$$\vec{a} \times \vec{b} = |a||b|\sin\varphi\ \hat{n} \qquad (3c)$$

where $\hat{n}$ is the unit vector perpendicular to the plane of vectors $\vec{a}$ and $\vec{b}$. Using this formula we see that equation (3a) is immediately recovered since the angle between the radial and momentum vectors is always π/2 by definition:

$$\vec{r} \times \vec{p} = |r||p|\sin\pi/2\ \hat{k} = rp = mvr\,\hat{k} \qquad (3d)$$

From which eq. (3b) directly follows.