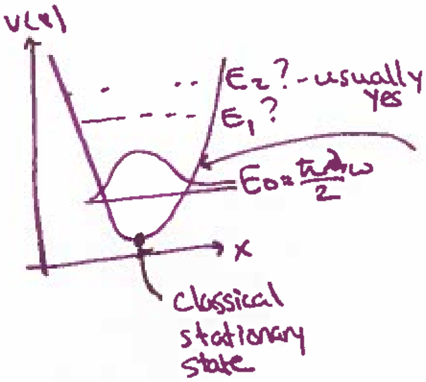
$\hat{H}\psi = i\hbar\frac{\partial \psi}{\partial t}$ has special solutions $\psi = \varphi(x)e^{-i\hbar E_n t} \Rightarrow P_n(x) = |\psi|^2$ is independent of time

$\varphi_n(x)$ must satisfy $\hat{H}\varphi_n = E_n\varphi_n$, where E_n is the energy of state φ_n with 100% probability



spring frequency: $\omega = \sqrt{\frac{k}{m}}$, $k = m\omega^2$

$\varphi_0(x) \sim$ gaussian $\sim e^{-m\omega x^2/2\hbar}$ if $m, \omega \neq 1$

There are a many more stationary states in QM

Worked Example:

Trick: are there operators "$\hat{a}^+$" and $\hat{a}$ such that

$\hat{a}^+\varphi_0(x) \sim \varphi_1 x$; $\hat{a}^+\varphi_1(x) \sim \varphi_2(x)$ } would be really nice if these operators exist (yes)

$\hat{a}\varphi_2(x) \sim \varphi_1(x)$; $\hat{a}\varphi_0(x) = 0$

Yes: $\hat{a}^+ = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(x - \frac{i}{m\omega}\hat{p}\right)$ and $\hat{a} = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(x + \frac{i}{m\omega}\hat{p}\right)$

Proof: $\hat{n} \equiv \hat{a}^+ a = \frac{m\omega}{2\hbar}\left\{\frac{1}{m^2\omega^2}\hat{p}^2 + x^2 + \frac{i}{m\omega}\underbrace{(x\hat{p} - \hat{p}x)}_{i\hbar}\right\}$

$\hat{n} \equiv a^+ a = \frac{1}{\hbar\omega}\left\{\underbrace{\frac{p^2}{2m} + \frac{\overset{k}{\cancel{m\omega^2}}}{2}x^2}_{\sim \text{Hamiltonian } \hat{H}} - \frac{\hbar\omega}{2}\right\}$, $m\omega^2 = k$

$\hat{n} \equiv \hat{a}^+\hat{a} = \frac{1}{\hbar\omega}\hat{H} - \frac{1}{2} \quad \Rightarrow \hat{H} = \hbar\omega\left(\hat{n} + \frac{1}{2}\right)$

product of raising & lowering operator

Next proof: just like $x\hat{p} - \hat{p}x = i\hbar \neq 0 \Rightarrow \underbrace{\hat{H}\hat{a}^+ - \hat{a}^+\hat{H} = \hbar\omega\hat{a}^+}_{\text{your hw}}$

Let $\varphi_n(x)$ be an eigenstate of $\hat{H}$ with eigenvalue E_n: $\hat{H}\varphi_n = E_n\varphi_n$

$\Rightarrow \hat{H}a^+\varphi_n = \hat{a}^+\hat{H}\varphi_n + \hbar\omega\hat{a}^+\varphi_n$

$= \hat{a}^+E_n\varphi_n + \hbar\omega\hat{a}^+\varphi_n$

$\hat{H}(\hat{a}^+\varphi_n) = (E_n + \hbar\omega)(\hat{a}^+\varphi_n)$

$\hat{H}\varphi_{n+1} = E_{n+1}\cdot\varphi_{n+1}$

So,

n	E_n	$\varphi_n(x)$
0	$\hbar\omega(\frac{1}{2})$	$\varphi_0 \sim e^{-m\omega x^2/2\hbar}$
1	$\hbar\omega(1+\frac{1}{2})$	$\varphi_1 \sim \hat{a}^+\varphi_0 = ?$
2	$\hbar\omega(2+1/2)$	$\varphi_2 \sim \hat{a}^+\varphi_1 = ?$
⋮		
n	$\hbar\omega(n+1/2)$	$\varphi_n \sim \hat{a}^+\varphi_{n-1} = ?$
↑ quantum number	energy levels of a harmonic oscillator	

example $\varphi_1(x) \sim \hat{a}^+\varphi_0(x)$

$$\sim \left(x - \frac{i}{m\omega}\hat{p}\right)\varphi_0(x)$$

$$\sim \left(x - \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\varphi_0(x)$$

$$\sim x e^{-m\omega x^2/2\hbar} - \frac{\hbar}{m\omega}\left(-\frac{m\omega}{\hbar}\right) x e^{-m\omega x^2/2\hbar}$$

or

$$\boxed{\varphi_1(x) \sim x e^{-m\omega x^2/2\hbar}}$$

only this energy can be absorbed $\Delta E = \hbar\omega$