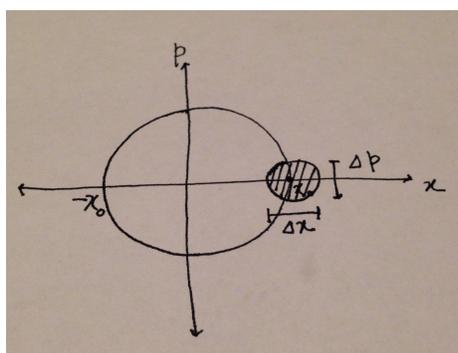


Hour Exam 1 Solution

1a. (5 pts) On an x-p plot, draw the trajectory of a ball vibrating on a spring, shown on the right. Draw the trajectory for the ball starting at $x_0 > 0$ and $p=0$. ($x=0$ is the equilibrium position of the spring.) **Draw one cycle of the vibration.** We started the plot for you below:

b. (10 pts) In quantum mechanics, a particle has a spread Δx and Δp associated with it. **Draw a quantum particle** on the same plot when its average momentum is $p=0$, and label Δx and Δp . **What is the “area”** of the quantum particle on the x-p plot? **Write down** the formula for the “uncertainty” principle involved.

Solution:



We assume the spring to be an ideal one, hence the classical particle comes back exactly to its initial position.

The quantum particle lies on the x axis, since its **average** momentum is zero. However, this does not mean its momentum has no uncertainty, since that would imply its position to have infinite uncertainty. The area of the quantum particle, which is the product of uncertainties in position and momentum, is given by Heisenberg's principle, $\Delta x \Delta p = \frac{\hbar}{2}$.

2. A wavefunction is given by $\psi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} e^{ikx}$, $k = 10 \text{ m}^{-1}$

a. (5 pts) **What is its numerical value** at $x=1$, written in the form $\Psi(x) = a + i b$?

b. (5 pts) **What is** the complex conjugate of the number $a + i b$ in general, and **what is** the complex conjugate of the wavefunction in (a) numerically ?

c. (5 pts) **Show that** the wavefunction times its complex conjugate is a real number.

Solution:

$$a. \quad \psi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} \{\cos kx + i \sin kx\} = \frac{1}{\sqrt{\pi}} e^{-x^2} \{\cos 10x + i \sin 10x\}$$

$$\therefore \psi(1) = \frac{1}{\sqrt{\pi}} e^{-1} \{\cos 10 + i \sin 10\} = -0.17 - 0.11i$$

b. The complex conjugate of $a + ib$ is $a - ib$. Therefore, $\psi^*(1) = -0.17 + 0.11i$.

$$c. \quad \psi(x) * \psi^*(x) = \frac{1}{\pi} e^{-2x^2} e^{ikx} e^{-ikx} = \frac{1}{\pi} e^{-2x^2}$$

Specifically,

$$\psi(1) * \psi^*(1) = \frac{1}{\pi} e^{-2} = 0.043$$

3. A pianist plays a scale of notes in a Chopin Ballade *prestissimo* (200 notes per second). He hits a high “A” (1760 Hz) for $1/200^{th}$ of a second.

- (5 pts) As a percentage of frequency, **what is** the spread in frequency of the high “A”?
- (5 pts) The next whole note on the piano is B, a factor of $2^{(2/12)} \approx 1.122$ higher in frequency than A. **What is** the frequency of B? Can A and B **be distinguished** when played $1/200^{th}$ second each?
- (5 pts) If a piano-playing robot were built that plays the passage at $1/1000^{th}$ second per note, **could** individual notes A, B, C... in a scale be distinguished?

Solution:

$$\begin{aligned} \text{a. We know, } \Delta\nu\Delta t &= \frac{1}{4\pi}, \Delta t = \frac{1}{200} \\ &\therefore \Delta\nu = \frac{200}{4\pi} = 15.92 \text{ Hz.} \\ &\therefore \frac{\Delta\nu}{\nu} * 100\% = \frac{15.92}{1760} * 100\% = 0.904\% \end{aligned}$$

$$\text{b. } \nu_B = 1.122 * \nu_A = 1.122 * 1760 \text{ Hz} = 1974.72 \text{ Hz}$$

The difference in frequency between A and B notes is clearly greater than the uncertainty in frequency (15.92 Hz). Hence, A and B can be distinguished.

$$\begin{aligned} \text{c. Now, } \Delta t &= \frac{1}{1000} \\ &\therefore \Delta\nu = \frac{1000}{4\pi} = 79.58 \text{ Hz} \end{aligned}$$

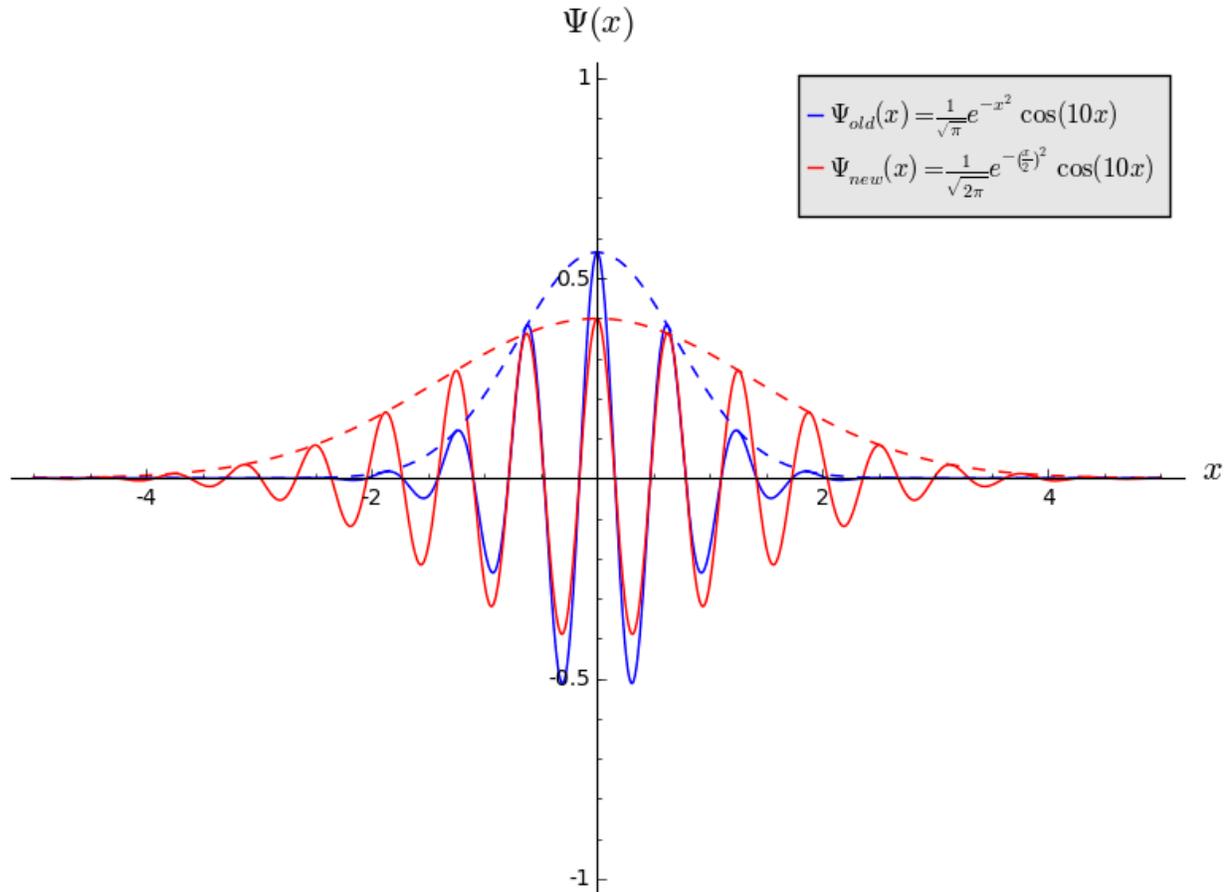
Clearly, the uncertainty in frequency is still lower than the difference in frequency of the notes (~ 214 Hz). Hence the notes can still be distinguished.

4. Below is shown a plot of the real part of the wavefunction $\Psi(x)$ in problem 2, and its Fourier transform $\Psi(k)$.

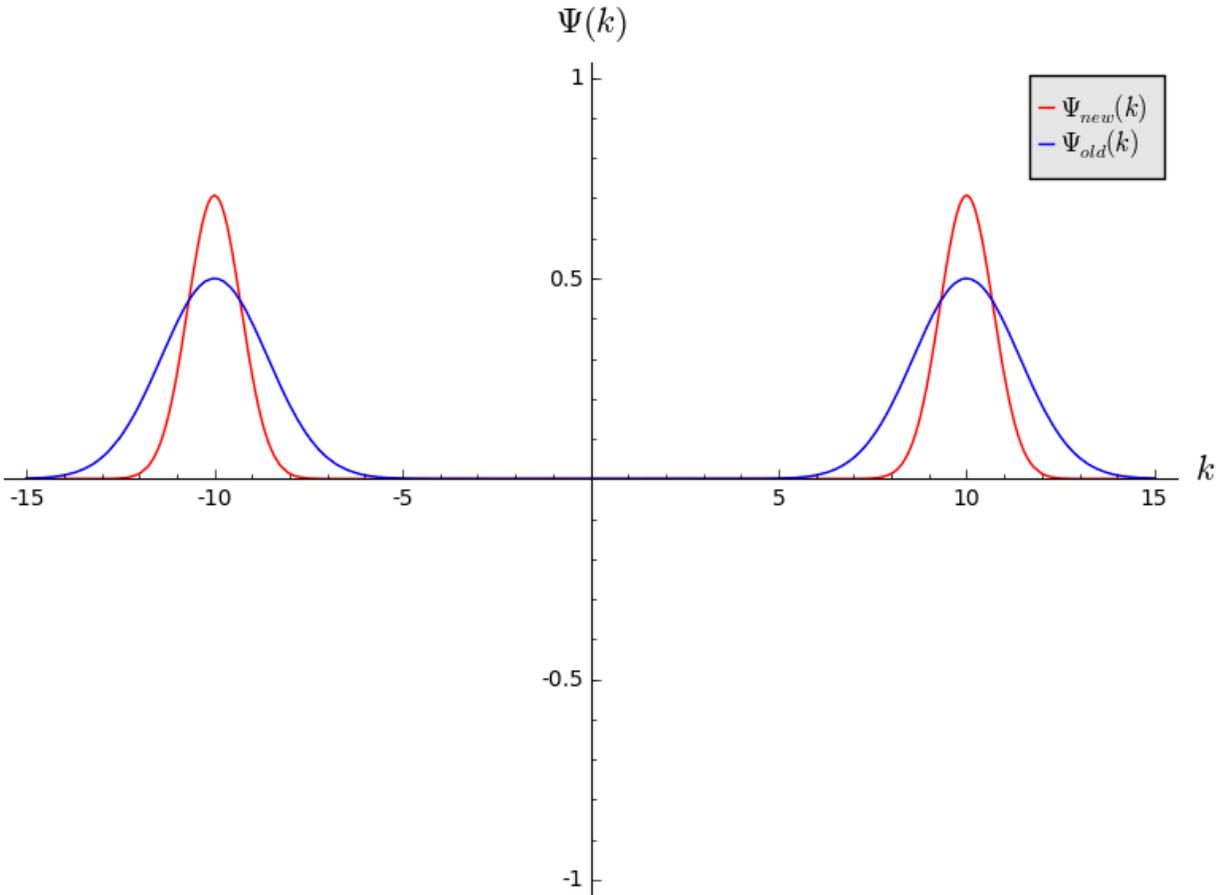
a. **Sketch roughly** the function $\Psi(x) = \frac{1}{\sqrt{2\pi}} e^{(-\frac{x}{2})^2} \cos 10x$ below it, and also **sketch its Fourier transform**. (The two functions differ by x vs. $x/2$ in the Gaussian.)

We break the function into two parts: Gaussian (e^{-x^2}) and $\cos 10x$ to understand the plot. The **Gaussian part forms the envelope of $\Psi(x)$** (denoted by the dashed lines in the plot below). The

cos 10x part forms the modulations. Therefore, we can analyze the changes in the two parts to understand the new plot. In the new wavefunction, **cos 10x remains the same**, indicating that the frequency of modulation is the same. However, the Gaussian is changed to $e^{-\left(\frac{x}{2}\right)^2}$, indicating that the width of the envelope should **broaden by a factor of 2** (recall that the width information is the denominator of the exponent). Also, the **amplitude becomes smaller because of the reduced leading coefficient**.



Since $\Delta x \cdot \Delta k = \frac{1}{2}$ (Result of Fourier Transformation of a Gaussian in x-space to k-space), **Δk is thus more narrow by a factor of 2, as shown.**



5. (5+5+5=15 pts) According to postulate (5), the wavefunction satisfies the two equations

$$\begin{aligned}
 + \frac{\partial \Psi^r}{\partial t} &= \frac{\hat{H}}{\hbar} \Psi^i \\
 - \frac{\partial \Psi^i}{\partial t} &= \frac{\hat{H}}{\hbar} \Psi^r
 \end{aligned}$$

from which we derived the time-dependent Schrödinger equation in class.

a. Replace the time derivative in both equations by a difference involving $\Psi^{r \text{ or } i}(t + \Delta t)$, $\Psi^{r \text{ or } i}(t)$ and Δt , and **write down** the resulting two finite-difference equations.

b. Solve the two equations for $\Psi(t + \Delta t)$ to **obtain**

$$\begin{aligned}
 \Psi^r(t + \Delta t) &= (1) \\
 \Psi^i(t + \Delta t) &= (2)
 \end{aligned}$$

with explicit equations in place of (1) and (2).

c. Now assume we get even lazier than baby calculus and on the computer, after we calculate $\Psi^r(t + \Delta t)$ from $\Psi^r(t)$ in the first line (1), we store the new value $\Psi^r(t + \Delta t)$ ON TOP OF the old value $\Psi^r(t)$, erasing it. **How does that modify** (2)?

Congrats, you have derived the SUR algorithm for moving the wavefunction forward in time on a computer.

- a. The definition of a derivative is nothing more than a slope, or “rise over run”, they say. Note that I have omitted the limit because Δt essentially cannot be infinitely small due to limitation of computer bits.

$$\begin{aligned}
 + \frac{\partial \Psi^r}{\partial t} &= \frac{\Psi^r(t + \Delta t) - \Psi^r(t)}{\Delta t} = \frac{\hat{H}}{\hbar} \Psi^i \\
 - \frac{\partial \Psi^i}{\partial t} &= - \frac{\Psi^i(t + \Delta t) - \Psi^i(t)}{\Delta t} = \frac{\hat{H}}{\hbar} \Psi^r
 \end{aligned}$$

- b. Rearranging the equations:

$$\begin{aligned}
 \frac{\Psi^r(t + \Delta t) - \Psi^r(t)}{\Delta t} &= \frac{\hat{H}}{\hbar} \Psi^i \\
 \Psi^r(t + \Delta t) - \Psi^r(t) &= \frac{\Delta t}{\hbar} \hat{H} \Psi^i(t) \\
 \Psi^r(t + \Delta t) &= \Psi^r(t) + \frac{\Delta t}{\hbar} \hat{H} \Psi^i(t) = (1)
 \end{aligned}$$

$$\begin{aligned}
 \frac{\Psi^i(t + \Delta t) - \Psi^i(t)}{\Delta t} &= - \frac{\hat{H}}{\hbar} \Psi^r \\
 \Psi^i(t + \Delta t) - \Psi^i(t) &= - \frac{\Delta t}{\hbar} \hat{H} \Psi^r(t) \\
 \Psi^i(t + \Delta t) &= \Psi^i(t) - \frac{\Delta t}{\hbar} \hat{H} \Psi^r(t) = (2)
 \end{aligned}$$

- c. If the computer executes line 1 first, overwriting the value $\Psi^r(t)$ by $\Psi^r(t + \Delta t)$ then when it executes the second line of code, equation (2) really becomes:

$$\Psi^i(t + \Delta t) = \Psi^i(t) - \frac{\Delta t}{\hbar} \hat{H} \Psi^r(t + \Delta t)$$

It turns out this is more accurate than the original eq. (2). Note that the code knows nothing about $\hat{H} \Psi^r(t + \Delta t)$ until another full cycle is evaluated.