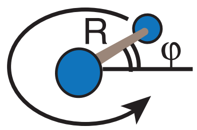
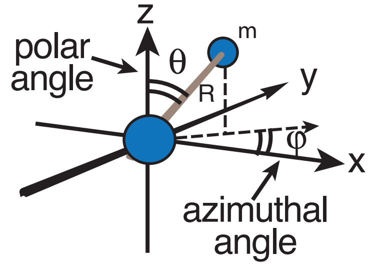
Rotation in two dimensions:
Only one angle needed:

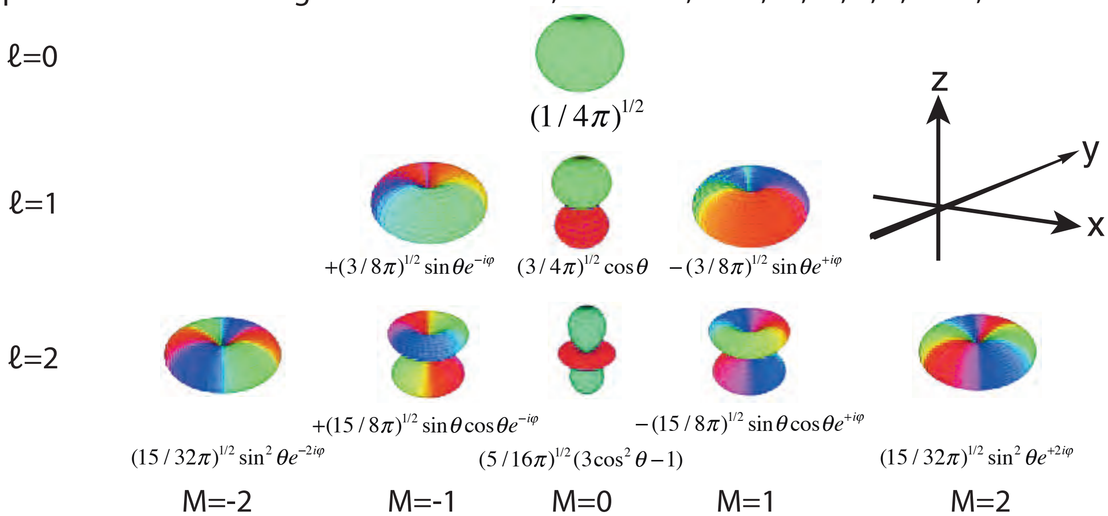


Rotation in three dimensions:
Two angles needed:



The $Y_{\ell M}$ functions solve the Schrödinger equation for rigid rotation in 3-D. M is the z-axis component of the total angular momentum ℓ, so M = -ℓ, -ℓ+1, ... , -1, 0, 1,... ℓ-1, ℓ+1

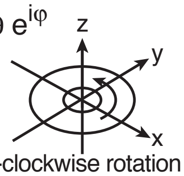
The color codes the complex phase, green=+1, red = -1, yellow and blue ar ±i.



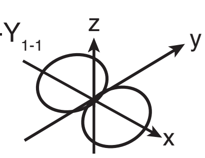
Since $E_{\ell,M}=\hbar^2\ell(\ell+1)/(2mR^2)$ is independent of M, all $Y_{\ell,M}$ with the same ℓ are degenerate, and we can make linear combinations as we please, including ones that are real functions:

ex: Y_{1-1} and Y_{11} are **degenerate**, so we **can combine them**

$Y_{11} \sim \sin\theta\, e^{i\varphi}$

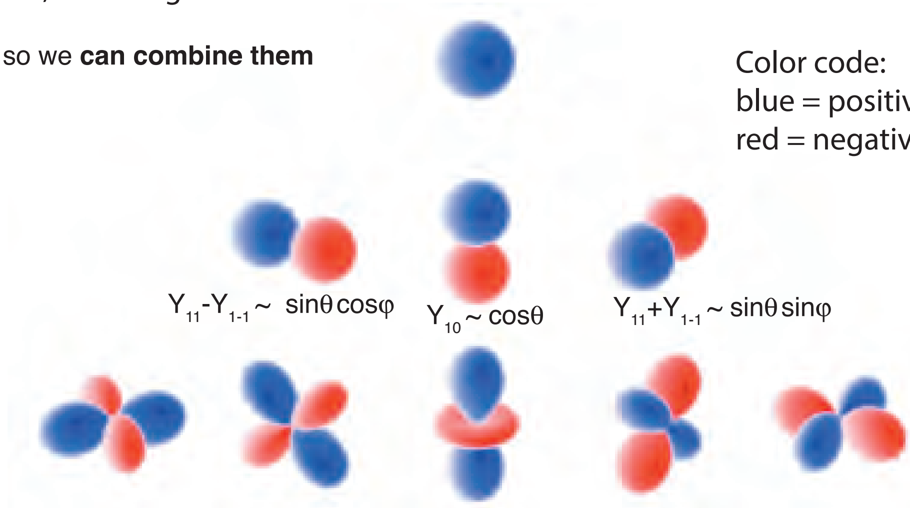


$Y_{11}-Y_{1-1}$



$Y_{11}-Y_{1-1} \sim \sin\theta\, e^{i\varphi}+\sin\theta\, e^{-i\varphi}$

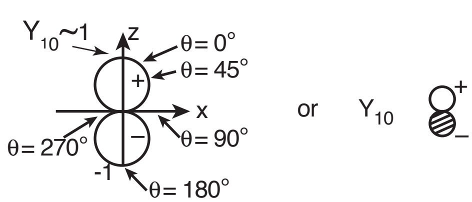
$\sim \sin\theta\cos\varphi$



Color code:
blue = positive
red = negative

How do we come up with these polar plots $R(\theta,\varphi) = Y_{\ell M}(\theta,\varphi)$?

$Y_{10} \sim \cos\theta$



$Y_{20} \sim 3\cos^2\theta - 1$