Homework H24 Solution

Turn in 1. What is the probability that a vibrating molecule in the n=0 state is in the tunneling region?

Solution: Recall that the energy of a vibrating molecule within the harmonic oscillator approximation is

$$E_0 = \frac{\hbar\omega}{2} \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}$$

The "classical turning point" of an oscillator occurs when its kinetic energy is zero and its potential energy is equal to its total energy. As you have seen previously, the potential energy is given by

$$V(x) = \frac{1}{2}kx^2$$

Therefore, the classical turning points are found by equating E_0 with V(x) and solving for x:

$$\frac{1}{2}kx^2 = \frac{\hbar\omega}{2}$$

$$x_{tp} = \pm\sqrt{\frac{\hbar\omega}{k}}$$

The tunneling region is the region beyond the classically allowed turning points, and we find the probability of the vibrating molecule being in this region in the usual way: by integrating the square modulus of the wave function over the domain of interest.

The ground state wave function for the harmonic oscillator is

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}$$

Thus we arrive at the expression for the probability of the vibrating molecule being in the tunneling region

$$P_T = 2\sqrt{\frac{m\omega}{\pi\hbar}} \int_{\sqrt{\frac{\hbar\omega}{k}}}^{\infty} e^{-\frac{m\omega x^2}{\hbar}} dx$$

where we have exploited the fact that the Gaussian is an even function to arrive at the factor of 2 in the expression for P_T.

Making the variable substitution $y = \sqrt{\frac{m\omega}{\hbar}}x$ yields (remember, $k=m\omega^2$)

$$P_T = 2\sqrt{\frac{1}{\pi}} \int_{1}^{\infty} e^{-y^2} dy$$

The integral can be solved numerically online or in Matlab, giving

$$P_T = 0.157299$$

So there is a nearly 1 in 6 chance the quantum spring will be found stretched or compressed to higher than its allowed energy! Note that you would never be able to conduct a measurement that violates energy conservation, though: measuring energy accurately requires time (e.g. to get a kinetic energy, you must measure where a particle is, wait, measure again, and divide Δx by Δt to get v and K=$^1/_2$mv^2, which takes time Δt). During that time Δt, the spring will spend the majority of its time in the classically allowed region, and all measurements of kinetic energy will come out with E > $\langle K \rangle$ > 0. In fact, one can prove a mathematical theorem (which Gruebele will spare you this time around!) that any measurement yields $\langle K \rangle = \langle V \rangle$ = E/2 >0 exactly and at all times for a harmonic oscillator! Quantum mechanics is devilishly clever in doing weird things, but always comes out looking like a harmless angel when it's down to the wire.

2. Worked Problem 4.3 (page 63)

Solution: It's in the book!

3. Problem 4.4 (Page 67)

Solution: The tunneling current, or rate of flow of tunneling electrons, is proportional to the probability that an electron will tunnel through the potential barrier. The probability of tunneling is given by,

$$P = exp\left\{-2L\sqrt{2m(V-E)}/\hbar\right\}$$

where L is the distance that the electron tunnels.

Now, $V - E$ = work function $= \phi = 4\ eV = 4 * 1.6 * 10^{-19} J = 6.4 * 10^{-19} J$

The distance between the metal tip and the conducting surface changes from 0.6 nm to 0.3 nm, when it passes over the conducting patch.

So, the factor by which the tunneling current increases is proportional to the ratio of probabilities at the two regions, given by,

$$\frac{P_1}{P_2} = exp\left\{-2 * \sqrt{2 * 9.1 * 10^{-31} * 6.4 * 10^{-19}} * (0.3 - 0.6)10^{-9}/(1.05 * 10^{-34})\right\}$$

$$= 464.296 \approx 465$$

Thus the tunneling current increases by a factor of 465. This sensitivity allows tunneling microscopes to measure surfaces with atomic accuracy. If you want to see one in action, email Gruebele, and you can see the one at the Beckman Institute, where quantum particles magically disappear and reappear on the other side of the barrier on a daily basis!