

# Notes S - Statistical mechanics and thermodynamics: from two simple assumptions to the free energy

## Chapter 1. From mechanics to statistical mechanics

Mechanics is very powerful. Equations such as  $F=ma$  or  $H\Psi=i\hbar\partial/\partial t\Psi$  allow us to calculate how a system of particles evolves in time. Both equations conserve energy for isolated system. In classical mechanics, the state of an isolated system is specified by the positions and momenta of all the particles. In quantum mechanics, the state is specified by a wavefunction, which depends only on the positions of the particles: you might think you need only half as many variables, but in reality, it's the same number because the wavefunction is a complex function (has a real and imaginary part) to make up for it. As we saw in quantum mechanics, the big difference between solving  $F=ma$  and solving  $H\Psi=E\Psi$  is that classical particles behave like points that can congregate arbitrarily closely, while quantum particles occupy an 'area'  $\Delta x\Delta p = \hbar/2$  - The Heisenberg Principle - and certain particles, called Fermions, can't share that area - the Pauli Exclusion Principle.

Very often, we don't want to know what every single particle does. For example, in a chemical reaction in a flask, there are many molecules reacting, and we may be interested only in what the equilibrium constant of the reaction is, or whether the reaction will go spontaneously or not.

Statistical mechanics makes the connection between microscopic dynamics and macroscopic dynamics. For example, while thermodynamics can manipulate equations of state such as  $PV=nRT$ , it cannot be used to derive them. Statistical mechanics can derive such equations and show how they arise from the world of atoms and molecules. **Our overall goal in chapters S1-6 is to start with two simple postulates in Chapter 2, and derive the equilibrium constant  $K$  and that the Gibbs free energy satisfies  $\Delta G < 0$  for a spontaneous chemical reaction, all from first principles. Then we'll have a quick look at how molecules move by diffusing.**

### Important definitions:

**Isolated system:** A collection of particles with their time-independent Hamiltonian  $H$  specified. ( $H$  may include constraints such as potential energy walls that confine the particles to a volume  $V$ .)

**Microstate:** A state of the system fully specified, i.e. the exact positions/momenta (classical), or the exact wavefunction (quantum) of all particles in the system are known.

**Macrostate:** A state of the system uniquely specified by macroscopic variables such as:  $E$  (total energy),  $V$  (volume),  $n$  (particle number),  $q$  (charge), etc.

**State functions:** The macroscopic variables that uniquely specify the macrostate of a system are called *state functions*.

**Reversible:** two macrostates that differ only by an infinitesimal change in state functions are said to be connected by a reversible process. A series of such infinitesimal transformations therefore is also reversible. **Irreversible** describes all other processes.

Example: A box filled with argon atoms is nearly an ideal gas. If you know where every atom is located and how fast it is moving, you know the microstate. If you know only average quantities like the energy  $E$  of the gas, volume  $V$  of the box, and total number of particles  $n$ , you know only the macrostate.

**Thought experiment:** If a typical human looked at a jar of red and blue marbles of equal luminance, and all the red ones were at the bottom, and another jar where they are mixed, would we call them different macrostates? Assume for a moment that we are dogs with gray-scale vision. Would we call them different macrostates? These definitions are not absolute, and observers need to agree what they consider 'macroscopic variables' before we can rigorously compare models and experiments!

For the first five chapters of statistical thermodynamics, we will compare only equilibrium states or states that differ by an infinitesimal change. Some of the equalities we derive for such reversible transformations would become inequalities for an irreversible process (a process that, even when sliced into infinitesimal steps, still contains steps with large changes – say an explosion, or suddenly removing a weight instead of gradually reducing the weight).

**IMPORTANT CONCEPT: Density or “probability density”**

The most fundamental object in statistical mechanics is the density. What is this mysterious ‘density’? It is simply the probability  $\rho_j$  that a macroscopic system will be in a specific microstate “j”. Statistical mechanics deals with this probability instead of every individual particle.

Three cases are common: discrete, classical and quantum system:

a. Discrete systems: an example would be rolling dice. The die sitting on the table is the macrostate. Which face points up is the microstate. A die has 6 faces, each of which is a “microstate.” Each outcome is equally likely if the die is fair, so

$$P = \rho_j = \frac{1}{W} = \frac{1}{6}$$

is the probability of being in microstate “j” where  $W=6$  is the total number of microstates and  $j$  ranges from 1 to 6, depending on which face of the die is pointing up.

**Homework Problem S1.1:** For two dice, what is the probability of being in macrostate “7”? What function does the probability distribution approach when I roll many dice at once in each roll, and add up the numbers? Make a plot, starting with the uniform distribution (dice have equal probability of 1/6 for each face) for  $N=2$  dice and  $N=20$  dice using

[http://195.134.76.37/applets/AppletCentralLimit/Appl\\_CentralLimit2.html](http://195.134.76.37/applets/AppletCentralLimit/Appl_CentralLimit2.html)



Figure: Here are all the microstates of two dice for being in macrostate “4”

b. Classical systems: here the microstate of an isolated system has to be specified by the positions  $x_i$  and the momenta  $p_i$  of all particles in the system, so

$$P(x_i, p_i, t) = \rho(x_i, p_i, t)$$

is the probability of finding the particles at positions  $x_i$  and with momenta  $p_i$  at time  $t$ .

c. Quantum systems: here the probability of microstate  $j$  of an isolated system is specified by the square of the wavefunction  $\psi_j$ , or

$$P(x, t) = |\psi_j(x, t)|^2.$$

In all three cases, statistical mechanics attempts to evaluate the probability  $\rho_j(t)$  from first principles, using the conserved energy (Hamiltonian) of the isolated system.

**In-class exercise:** For any observable  $A_j$  known in every microstate “j”, if we also know the probability  $\rho_j$ , what is the average value (expectation value) of the observable? [Hint: the answer is in the Overview - goals of chem chapter and in next chapter, if you need to look it up.]

**Homework problem S1.2:** A bag contains two 1 dollar coins and 4 quarters. If you draw a single coin from the bag many times in a row, what is the average value of a draw? Could you ever draw that amount in a single attempt? If the bag contained a dollar coin, a half dollar and 2 quarters, what would be the answer to the same two questions? Beware: experiments can, but do not always, give the 'average' value as a result when you do them!

## Chapter 2. Postulates of statistical mechanics, entropy, and thermodynamic laws

### 1. Postulates and the ergodic approximation

Thermodynamics puts constraints on the behavior of macroscopic systems *without* talking about the underlying microscopic properties. It does not provide a connection to the microscopic origin of its fundamental quantities  $E$  (energy) and  $S$  (entropy). For  $E$ , this is less of a problem because energy is a more intuitive quantity and we know from mechanics that

$$E = \sum \frac{1}{2m} p_i^2 + V(x_i),$$

and the macroscopic formula for  $E$  arises by integrating over most coordinates and velocities.

Statistical mechanics makes the macro-micro connection and provides a quantitative description of  $E$  and  $S$  in terms of microscopic quantities. For large systems (except near the critical point), its results are in agreement with thermodynamics: one can derive the laws of thermodynamics from statistical mechanics, and we will do so. For systems undergoing large fluctuations (small systems, or systems near a critical point), the predictions of statistical mechanics are different from thermodynamics and more accurate.

#### Important definitions:

An **ensemble** is a collection of all the microstates of a system consistent with the macrostate as specified by the system's state functions such as  $E, V, N, q, \dots$

A **partition function** of a system is the average number of microstates accessible to the system in a given macrostate

An **extensive** state function or variable scales linearly with the size of the system (e.g. total mass)

An **intensive** state function or variable is independent of the size of the system (e.g. density)

#### The goal of statistical mechanics:

- Given a system of many particles "i" with positions  $x_i$  and momenta  $p_i$  (or with wavefunction  $\Psi(x_i)$ , find the average value  $A$  of any observable  $A(x_i, p_i)$  (or of any operator  $\hat{A}$ ) as particles move about, averaged over the ensemble. At equilibrium, the average value  $A$  does not change with time.

The two main postulates of statistical mechanics are:

#### **Postulate I: The extension of microscopic laws to macroscopic systems**

Microscopic dynamics (e.g.  $F=ma$ ) applies to any finite isolated system whose macrostate is uniquely specified by its Hamiltonian and its extensive state functions ( $E, V, N, \dots$ ).

Here  $E$  is the energy,  $V$  the volume, and  $N$  the number of particles. You may need to specify other state functions, such as length  $L$  if the system is one-dimensional, but we'll use  $E, V, N$  as examples (or instead of  $N$ , mole number  $n = N / 6.02214076 \cdot 10^{23}$ , if  $N$  is inconveniently large).

#### **Postulate II: The principle of equal probabilities**

The principle of equal probabilities in ensemble (weak) form and time (strong) form is:

- Weak form: All  $W$  microstates "j" of a system satisfying Postulate I have equal probability  $\rho_j = 1/W$ . These  $W$  microstates are called the microcanonical ensemble.  $W(E, V, n, \dots)$  is called the microcanonical partition function; it counts the number of microstates in the macrostate of postulate I.
- Strong form: for any ensemble satisfying i) at equilibrium,  $\langle \rho_j \rangle_t = \langle \rho_j \rangle_E$  (ergodic principle). This states that averaging a single system over time is equivalent to averaging over the ensemble of all  $W$  microstates.

From these two postulates, we can calculate any average property  $A$  of the system in a macrostate by averaging over all its microstates. Thus for the microcanonical ensemble of  $W$  microstates “ $j$ ”, each of which has value  $A_j$  for the observable  $A$ , the average value or expectation value of  $A$  is simply, as we saw in our introduction to pchem,

$$A = \sum_j \rho_j A_j = \frac{1}{W} \sum_{j=1}^W A_j$$

(discrete probability density) or  $A = \int \rho(x_j, p_j) A(x_j, p_j) dx_j dp_j$  (continuous probability density). Note that we deal with averages like  $E$  or  $A$  so much that we will leave out the bar as in  $\bar{E}$ , whereas the unaveraged quantities will be labeled  $E(x, p)$  or  $E_j$ .

Before we use them, these postulates require some discussion.

I) This is a strong statement; the system often has  $>10^{20}$  coordinates, and we assume that the dynamics are the same as for a few particles!

In an “isolated system,” the Hamiltonian  $\hat{H}$  is independent of time (the energy  $E$  is constant). **Therefore the first law of thermodynamics is automatically satisfied.** In an isolated system, the volume is also constant, so is the particle number, etc. Thus we can write the microcanonical partition function  $W(E, V, n, \dots)$ , as a function of all the conserved quantities needed to describe our isolated system.

II-i) Classically, this says that two configurations ‘ $j$ ’ and ‘ $k$ ’ of a system with different values of  $x_i$  and  $p_i$  of the individual particles are equally likely to be occupied as long as the energy  $E$  and other specified state functions of the two configurations are the same:  $\rho_j = \rho_k$

Quantum-mechanically, this means that two states  $\Psi_j$  and  $\Psi_k$  with the same energy  $E$  are equally likely to be occupied.

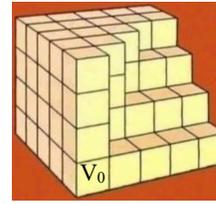
II-ii) This is also a strong statement: it says that we could take a single initial microstate of a system (with one a specific set of  $x_i$  and  $p_i$ ) and evolve it in time, eventually that system will visit all of the microstates at the same energy (and other state functions) as the initial microstate. This property is known as ‘ergodicity.’ In practice, ergodicity cannot really be satisfied, but we can use ii) for ‘all practical purposes’ instead of i) if the system visits enough typical microstates that the time average is not too different from the true ensemble average.

**Homework problem S2.1:** A measurement of the ammonia out-of-plane bending angle by the “Coulomb explosion technique” yields values  $-27^\circ, +16^\circ, +20^\circ, -12^\circ, +8^\circ, -12^\circ$ . Calculate the average by assigning a weight to each distinct value (e.g.  $1/6$  for  $+8^\circ$ ). Are any actual measurements near the average value? Remembering what you learned about tunneling in quantum mechanics, why are none of the measurements near the average?

**Homework problem S2.2:** How many distinguishable ways are there of arranging  $N=3$  different-colored balls in a box with three compartments? Draw all the configurations, and express your answer in terms of a factorials. What if all balls are blue and you can’t tell the difference between them, how many distinguishable ways are there then? Write your answer in terms of a ratio of factorials. What if 2 balls are red, and 1 is blue?

Example showing why ergodicity cannot be satisfied in reality:

We will use a discrete system to illustrate. Consider a box with  $M=V/V_0$  cells, filled with  $N \ll M$  particles of volume  $V_0$ . The dynamics is that the particles hop randomly to unoccupied neighboring cells at each time step  $\Delta t$ . This model is called a lattice-ideal gas. The number of arrangements for  $N$  identical particles is



$$W = \frac{M!}{(M-N)!} \cdot \frac{1}{N!}$$

Large factorials  $n!$  (or gamma functions  $\Gamma(n-1) = N!$ ) can be approximated by Stirling's formula

$$n! \sim n^n e^{-n}, \text{ or } \ln[n!] = n \ln[n] - n$$

Thus,

$$W = \frac{M!}{(M-N)!} \cdot \frac{1}{N!} \approx M^M (M-N)^{N-M} N^{-N} \approx \left(\frac{M}{N}\right)^N.$$

Let us plug realistic numbers into this:

$$V_0 = 10 \text{ \AA}^3, V = 1 \text{ cm}^3 \Rightarrow M = \frac{V}{V_0} = 10^{23}.$$

$$\text{For } N = 10^{19} \text{ gas molecules } (\sim 1 \text{ atm}) \Rightarrow \frac{M}{N} = 10^4$$

$$v_{\text{gas}} \approx \left(\frac{U_0}{m}\right)^{1/2} \approx 300 \text{ m/s } (\text{O}_2 \text{ at R.T.}) \Rightarrow \Delta t \sim \frac{L_0}{v} = \frac{V_0^{1/3}}{v} \sim 10^{-12} \text{ s} = 1 \text{ ps}$$

Lifetime of universe:  $\lesssim 10^{11} \text{ a} \sim 10^{18} \text{ s}$

$$W_{\text{possible}} \sim \frac{10^{18} \text{ s}}{10^{-12} \text{ s}} \sim 10^{30}$$

$$W_{\text{actual}} \approx (10^4)^{10^{19}} = 1 \text{ boogol} \gg 10^{30}$$

The possible  $W$  that can be visited during the lifetime of the universe is a mere  $10^{30}$ , negligible compared to the actual number of microstates  $W_{\text{actual}}$  at constant energy.

## 2. A calculation with $W$ and its properties

In principle, we can perform all statistical mechanical calculations if we know  $W$ , and thus the total number of microstates and the probability of a microstate.

**In-class exercise:** Consider again the example of particles filling cells in a box. Consider a box of volume  $2V$ , with  $2M$  cells of volume  $V_0$ . Initially, let all  $N$  identical particles be on the left side, so the right side of the box is empty. The particles hop randomly from cell to cell at each time step  $\Delta t$ . What is the probability that the particles remain on the left side of the box?

**Answer:** Initially, the number of microstates is  $W_{\text{before}} = W_{\text{left}} W_{\text{right}} = M! / [(M-N)! N!] \cdot 1$ , since there is only one way to arrange no particles in the right box. After the particles move around freely,  $W_{\text{after}} = (2M)! / [(2M-N)! N!]$ . Thus the probability that the particles remain on the left side is, if all microstates have equal probability (Postulate II)

$$\rho_{\text{left}} = \frac{W_{\text{left}}}{W_{\text{after}}} = \frac{W_{\text{before}}}{W_{\text{after}}} = \frac{M!(2M-N)!}{(2M)!(M-N)!}$$

If  $M$  and  $N$  are large (see example in a!), we can use Stirling's formula again. And

$$\begin{aligned} \rho_{\text{left}} &\approx \frac{M^M (2M-N)^{2M-N}}{(2M)^{2M} (M-N)^{M-N}} e^{-M-2M+N+2M+M-N} \\ &\approx \frac{M^N}{(2M)^N} = \frac{1}{2^N} \end{aligned}$$

For many particles, this is an incredibly small probability, so the particles will never stay on the left side. If they are free to move, they are going to fill the whole volume  $2V$ , and the chance that they will all spontaneously go back to the left side is  $2^{-N} = 2^{-10^{19}}$  for  $1 \text{ cm}^3$  of gas particles at 1 atmosphere.

No matter what our system looks like, it is clear that when particles are free to move, this always produces an increase in  $W$ , the number of available microstates. If the number of particles is large,  $W$  increases astronomically when the constraint is removed, scaling as  $e^N$ . Thus we can deduce in general the following law for  $W$  whenever the number of particles is large enough, Postulate II holds, and a constraint is removed at  $t=0$ :

$$W(t>0) > W(t=0).$$

The more the particles have a chance to move around, the more disordered the system is. Thus  $W$  is a measure of disorder: disorder is the quantity that always increases when constraints are removed. Another way of looking at it, in terms of time (Postulate 2 (ii)): if the system is initially in a 'rare' microstate (e.g. all particles on the left side), then it quickly evolves into a 'typical' microstate (e.g. particles everywhere in the box) and will stay in 'typical' microstates because there are so many more 'typical' than 'rare' microstates. At best, if the system is already in a 'typical' microstate to begin with, we can have the equality  $W(t>0) = W(t=0)$ .

### 3. $W$ and the entropy

However,  $W$  has a property that is not very desirable from a thermodynamic point of view. Recall that in thermodynamics, we prefer to work with intensive or extensive quantities: intensive quantities are independent of the size of the system. Extensive quantities increase linearly with the size of the system. Consider two identical systems from our microcanonical ensemble, each at energy  $E_1=E_2=E$ , and each with  $W_1=W_2=W$  possible microstates. Clearly for two isolated systems combined into a new isolated system,  $E_{\text{total}}=E_1+E_2=2E$  because energy is conserved (Postulate I, from which follows the first law of thermodynamics). The total energy is twice the energy. But  $W_{\text{total}}=W_1 \cdot W_2$ . For example, if there are 10 arrangements of the particles in each system, there are  $10 \cdot 10=100$  arrangements of the particles in the two systems. Thus  $W$  is neither an intensive quantity, nor an extensive quantity. It is a multiplicative quantity.

We can easily fix this: any function  $c f(W)$  that increases monotonically with  $W$  will also satisfy

$$c f(W(t>0)) \geq c f(W(t=0)),$$

where  $c$  is a positive constant. In particular, the logarithm is monotonically increasing AND has the property  $\ln(a \cdot b) = \ln(a) + \ln(b)$ . The logarithm turns multiplicative functions into additive functions.

We now define the new function

$$S = k_B \ln(W),$$

(so we let  $c=k_B$ ). It is then always true that

$$S(t>0) \geq S(t=0).$$

The equality will hold if the system is already in a 'typical' microstate at  $t=0$ , a situation we call 'equilibrium.'

**This is known as the second law of thermodynamics, and the new function is called the entropy.**  $S$  is an extensive function. Like  $W$ , the entropy also measures the disorder of a system, but in an extensive (additive) way. Note that we do not need to work with  $S$ . We could always calculate  $W$  to solve any problem in statistical mechanics, such as what the average value of any observable  $A$  is. But the entropy is very convenient: it is an extensive function, and it always increases in time for an isolated system whose energy is conserved.

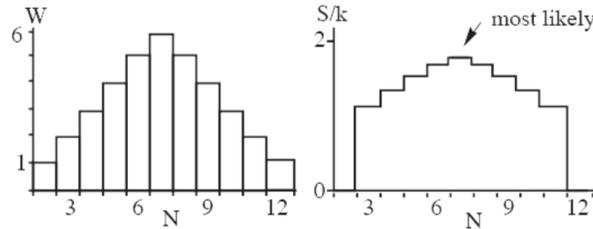


Figure:  $W$  for rolling two dice ( $N$  can go from 2 to 12), and entropy.  $W$  goes from 1 to 6, the entropy only from 0 to  $<2k_B$ . The difference is much more dramatic when  $W=10^{20}$ .

Now a word about the units of entropy. In other fields, like information science, entropy is defined with  $c=1$ . As you can see, the entropy defined above has units of  $k_B=1.38 \cdot 10^{-23}$  J/K. If chemists of the 19<sup>th</sup> century had understood that temperature is just the average energy per active degree of freedom of a system, the entropy would be unitless because temperature and energy would have the same units of Joules. But in the 19<sup>th</sup> century, people were confused about what temperature really is, coming up with ‘bad’ scales (like  $^{\circ}\text{C}$ , which allows negative values of  $T$ ) because they did not know that there is a minimum temperature reached when the average available energy per active degree of freedom goes to zero (all energy is extracted from the system). So in chemistry, temperature and energy have different units, and the ratio of these units is given by Boltzmann’s constant  $k_B$  ( $\approx 1.38 \cdot 10^{-23}$  J/K). As we will see later, so that  $TS$  has units of energy, we thus define  $S=k_B \ln(W)$ .

We already know that the 2<sup>nd</sup> law of thermodynamics that we derived above is independent of the choice of the constant  $c$ , as long as  $c>0$  such that  $S(t>0) > S(t=0)$ . As long as  $c$  is a positive number,  $W$  can be recovered uniquely from  $S$  because the logarithm increases monotonically with its argument, and  $S$  increases when constraints are removed.

**Homework problem S2.3:** Using the boxed example two pages ago, show that  $S \approx Nk_B \ln(V/V_0) - Nk_B \ln N$  if  $M$  (number of volume elements  $V_0$  in the box) is much larger than  $N$  (the number of particles). If I double the size of a box containing  $6.2 \cdot 10^{20}$  molecules (roughly 1 mMole), by how much does the entropy increase in units of J/K? What if I increase the volume by a factor of 1000? As you can see, entropy grows very slowly with volume, but it does increase because more volume  $\Rightarrow$  more arrangements  $\Rightarrow$  more disorder is possible.

We can also write the formula above in a slightly different way that can be more useful for general calculations:

$$S = k_B \ln(W) = -W \cdot \frac{1}{W} \ln\left(\frac{1}{W}\right)$$

$$S = -\sum_{i=1}^W \frac{1}{W} \ln\left(\frac{1}{W}\right)$$

$$S = -\sum_{i=1}^W \rho_i \ln(\rho_i)$$

This last formula writes the entropy explicitly in terms of the probability of each microstate. It could also be applied in cases where the probability of states in the sum is not the same (although it is required to be the same by postulate II in the microcanonical ensemble).

Finally, consider the case where all possible energy has been removed from the system. It can be proved from Postulate I that the ground state of any Hamiltonian of the type

$$H = \sum_i \frac{p_i^2}{2m_i} + V(x_i)$$

where  $V$  is finite at finite  $x$  is singly degenerate. Thus when  $E$  approaches its minimum value  $\Rightarrow W=1$ , there can be only one microstate. Clearly, we must then have  $S = k_B \ln(1) = 0$ . The entropy of a system approaches 0 when all the available energy is extracted from the system:

$$\lim_{E \rightarrow \min} S = 0.$$

**This is known as the third law of thermodynamics.**

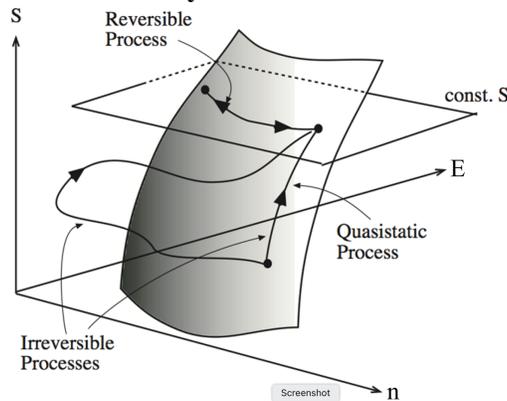


Figure: Entropy  $S$  increases monotonically as energy increases, i.e.  $\partial S / \partial E > 0$  always. Only processes at constant entropy are reversible according to the second law  $S(t > 0) > S(t = 0)$ . A quasistatic process is one done slowly enough so the system's trajectory lies on an 'equation of state' surface  $S = S(E, V, n)$  that specifies what  $W$  (and thus  $S$ ) is at a given energy, volume, particle number, etc. of the system.

Thus the postulates of statistical mechanics secure for  $S$  all the properties in the second and third laws of thermodynamics, and for  $E$  in an isolated system the first law of thermodynamics (energy conservation). We thus have now proved the laws of thermodynamics based on the microscopic properties of the Hamiltonian and the assumption of equal probability of states of equal energy. Note that statistical mechanics only has two postulates, one fewer than thermodynamics, and that we can prove the postulates of thermodynamics in the framework of statistical mechanics. This tells us that statistical mechanics is a more generally valid theory than thermodynamics, like quantum mechanics is more generally valid than Newtonian mechanics.

### Chapter 3. Entropy, temperature, and the microcanonical partition function: how to calculate results with statistical mechanics.

The goal of equilibrium statistical mechanics is to calculate the ‘density’ (probability)  $\rho_{eq}$  such that we can evaluate average observables  $A$  that give us fundamental relations or equations of state.

Just as thermodynamics has its potentials  $E$ ,  $H$ ,  $F$ , and  $G$ , so statistical mechanics has its ensembles. So far we only considered the microcanonical ensemble. The microcanonical ensemble is the one directly defined in postulate II of statistical mechanics. In the microcanonical ensemble  $E$  is fixed (Postulate I), and the other constraints that are fixed are the volume  $V$  and mole number  $n$  (for a simple system), or other extensive parameters such as charge  $q$  (for more complicated systems).

#### 1. How do we define temperature (and other variables) intuitively?

**a. Monotonicity of  $S(E)$  and  $E(S)$ .** We saw in chapter 2 that  $S=k_B \ln(W(E,V,n,\dots))$ , and that  $W$  increases monotonically with  $U$ : the higher the energy, the more microstates are accessible to the system. But the logarithm is also a monotonically increasing function, so  $S(E,V,n)$  is a monotonically increasing function of  $E$ : as the energy goes up, so does the entropy (or disorder) of a system because more microstates are available. More microstates means more disorder. Only 1 microstate means complete order ( $S=0$ ), as we saw in chapter 2. We can thus invert the function uniquely, and  $E(S,V,n)$  is a monotonically increasing function of  $S$ : the more disorder a system has, the higher its energy must be:

$$\left(\frac{\partial E}{\partial S}\right)_{V,n,\dots} \geq 0$$

**b. Derivatives of the energy and entropy.** If the energy changes a little bit, we have the differential change

$$\begin{aligned} dE &= \left(\frac{\partial E}{\partial S}\right)_{V,n} dS + \left(\frac{\partial E}{\partial V}\right)_{S,n} dV + \left(\frac{\partial E}{\partial n}\right)_{S,V} dn + \dots \\ &= T dS + (-P) dV + \mu dn + \dots \end{aligned}$$

The derivatives are intensive variables (independent of system size). Solving for  $dS$  we also have

$$dS = \frac{1}{T} dE + \frac{P}{T} dV - \frac{\mu}{T} dn + \dots, \text{ so } \left(\frac{\partial S}{\partial E}\right)_{V,n} = \frac{1}{T}, \left(\frac{\partial S}{\partial V}\right)_{E,n} = \frac{P}{T}, \text{ etc.}$$

Note that we are giving these derivatives names, like  $T$  and  $-P$  and  $\mu$ , but we usually reserve these names for temperature and pressure and chemical potential. Are they actually the temperature and pressure and chemical potential?! Let’s find out.

What is this quantity  $T$ ? Consider an isolated composite system {Sys} made of two subsystems {Sys<sub>1</sub>} and {Sys<sub>2</sub>} separated by a diathermal wall. A diathermal wall allows only energy to flow back and forth in the form of heat, so  $dV=0$  and  $dn=0$ . At equilibrium, the number of microstates of the composite system {Sys} is not changing, so

$$dS = 0 = \left(\frac{\partial S_1}{\partial E_1}\right)_{V,n} dE_1 + \left(\frac{\partial S_2}{\partial E_2}\right)_{V,n} dE_2$$

or

$$dS = \frac{1}{T_1} dE_1 + \frac{1}{T_2} dE_2.$$

But  $dE = 0$  for an isolated system because of energy conservation, from which follows that

$$dE_2 = -dE_1 \text{ or } 0 = \left(\frac{1}{T_1} - \frac{1}{T_2}\right) dE_1 \text{ at equilibrium.}$$

This equation can only be true if

$$\left(\frac{1}{T_1} - \frac{1}{T_2}\right) = 0 \text{ and therefore } T_1 = T_2$$

Thus,  $T$  is the intensive quantity that is equalized between two subsystems when heat is allowed to flow between them.

**Thought experiment:** Intuitively, what thing is the same between two objects if heat does not flow between them when in contact? For example, if I have two glasses of water and hold them together, but no heat flows from one glass to the other, what can we say about the water in the two glasses.

That's the definition of temperature: the thing that becomes equal when heat stops flowing between two places. We can thus identify the intensive variable  $T$  as the temperature of the system. Temperature is always guaranteed to be positive because we saw in **a.** that  $E$  is a monotonically increasing function of  $S$  and  $T = (\partial E / \partial S)$ .

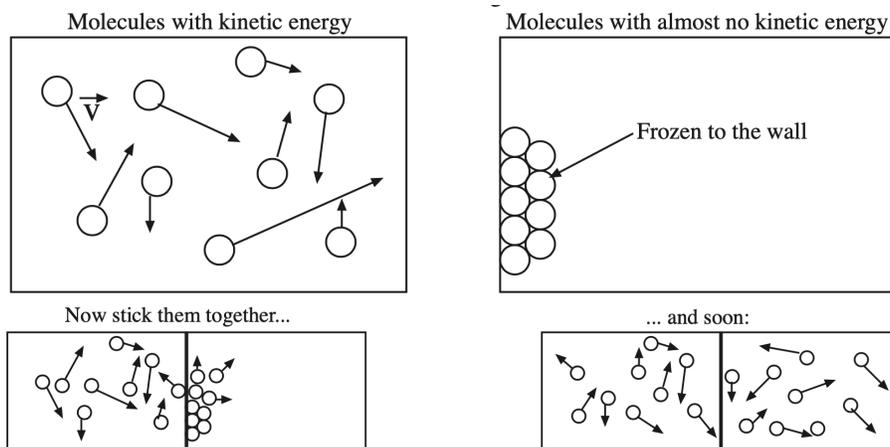


Figure: molecules exchange kinetic energy through collisions when in contact: heat flows (from left to right in our example). Once the energy per molecule has been equalized on averaged heat no longer flows. We say ‘the temperature is the same on both sides.’ Temperature difference acts like a driving force for heat flow.

**Homework problem S3.1:** Use the same logic to prove that  $P$  must be the pressure. **Pressure is the quantity that is equalized when two systems connected by a movable wall reach equilibrium by changing their volumes.** Note that the derivative  $\partial E / \partial V$  is defined as  $-P$  to agree with our intuition that volume decreases ( $dV$  is negative) by squeezing on the system, its energy increases ( $dE$  is positive) because we are doing work on the system.

**Homework problem S3.2:** Now prove that  $\mu$ , the chemical potential, is the quantity that is equalized between two subsystems when particles flow back and forth (or react) and their particle numbers come to equilibrium. You could go on and on proving such relationships for other intensive (derivative) variables.

We call pairs of variables such as  $S$  and  $T$ , or  $P$  and  $V$ , or  $\mu$  and  $n$  ‘thermodynamically conjugate state functions,’ or simply ‘conjugate.’ Like the Fourier-conjugate variables in quantum mechanics, they are mutually exclusive, in the sense that a thermodynamic state function cannot be a function of both of them simultaneously, e.g.  $E = E(S)$ , not  $E(S, T)$  because  $T$  is just the derivative  $\partial E / \partial S$  and thus not independent of  $E$  and  $S$ . Each conjugate pair contains an extensive variable (e.g.  $S$ ) that depends on system size, and an intensive variable (e.g.  $T$ ) that is independent of system size. As we will see in section 2., it is possible to replace  $E(S)$  by another function  $f(T)$  that contains the same information.

**c. An explicit formula for the energy.** In **b.**, we are giving a formula for energy change. Can we derive one for the energy  $E(S, V, n, \dots)$  explicitly? Since  $E, S, V, n, \dots$  are all extensive quantities, it must be true for any positive multiplier  $\lambda$  that

$$E(\lambda S, \lambda V, \lambda n, \dots) = \lambda E(S, V, n, \dots)$$

This simply states if I increase the size of the system by  $\lambda$ , so  $S \rightarrow \lambda S$  and so forth, the new energy must equal  $\lambda E$ . Taking the derivative with respect to  $\lambda$  on both sides,

$$\left(\frac{\partial E}{\partial \lambda S}\right) \left(\frac{\partial \lambda S}{\partial \lambda}\right) + \left(\frac{\partial E}{\partial \lambda V}\right) \left(\frac{\partial \lambda V}{\partial \lambda}\right) + \dots = E(S, V, n, \dots) \text{ or}$$

$$\left(\frac{\partial E}{\partial \lambda S}\right) S + \left(\frac{\partial E}{\partial \lambda V}\right) V + \dots = E(S, V, n, \dots).$$

This must hold for  $\lambda=1$  in particular, so

$$\left(\frac{\partial E}{\partial S}\right) S + \left(\frac{\partial E}{\partial V}\right) V + \dots = E(S, V, n, \dots) \text{ or}$$

$$TS - PV + \mu n + \dots = E(S, V, n, \dots)$$

This remarkably simple and universally true formula says that energy is simply a sum over the product of conjugate variable pairs that describe the macrostates of our system. It is known as ‘bilinear form’ or ‘Euler’s form’ of the energy in terms of intensive and extensive thermodynamic state functions. It is always true, no matter how complicated the system!

Systems that are more complicated than a gas need additional pairs of conjugate variables added: for a spring extended by length  $L$  when a force  $F$  is applied, a “ $-FL$ ” term is added to the energy instead of “ $-PV$ ”; for a soap bubble of surface area  $A$  and surface tension  $\sigma$ , a term “ $-A\sigma$ ” has to be added. And so on, but the energy is always ‘bilinear’ in terms of thermodynamic variables, and this is not an approximation.

**Thought experiment:** Other systems may of course have other variables besides  $V, n$ , etc. What about a protein of charge  $q$  in an electric potential  $V_e$ ?

## 2. Thermodynamic potentials $F, G$ , and $H$ are equivalent to $W, S$ , and $E$

We started with postulate II about  $W(E, V, n, \dots)$ , and showed that the functions  $S(E, V, n, \dots)$  (basically the log of  $W$ ) and  $E(S, V, n, \dots)$  contain the same information, but may be more convenient because the logarithm makes them less humongous. More such functions, or “thermodynamic potentials,” can be devised that also contain the same information as  $E$ .

**Thought experiment:** (see Figure) Let’s say we want  $E$  as a function of temperature, not entropy. Why can’t we just calculate  $T(S)=\partial E/\partial S$ , solve for  $S(T)$ , plug into  $E(S)$  to get  $E(T)$ ? That would give us  $E$  as a function of  $T$ , wouldn’t it? Answer: We lose info about the original function  $E(S)$ .

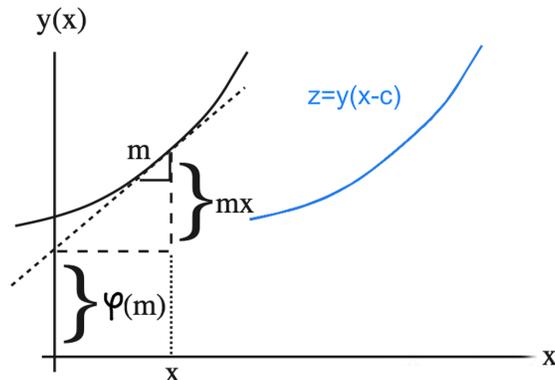


Figure: Changing  $y(x)$  to the equivalent function intercept(slope) =  $\phi(m)$ . Note that you cannot uniquely define  $y(m)$ , you really need  $\phi(m)$  to replace  $y(x)$ : for example, the function  $z(x)=y(x-c)$  is different from  $y(x)$ , but produces the same  $z(m)=y(m)$ , so  $z(m)$  and  $y(m)$  cannot be used to reconstruct the original and different  $y(x)$  and  $z(x)$ .

**Homework problem S3.3:** Illustrate this for the two functions  $y(x)=x^2$  and  $z(x)=(x-5)^2$ . First, calculate  $m(x)=\partial y/\partial x$ , solve for  $x(m)$ , and insert back into  $y$  to get  $y(m)=?$ . Then do the same thing for  $z(m)$ . The functions  $y(m)$  and  $z(m)$  are identical. So no way, if I give you  $y(m)$  or  $z(m)$ , can you tell me what the original different functions  $y(x)$  and  $z(x)$  were.

So how do we do this right? Consider a function  $y(x)$ . We can calculate its slope  $m(x)=\partial y/\partial x$  and its intercept  $\varphi=y(x)-mx$ , and then express the intercept as a function of slope,  $\varphi(m)=y(x(m))-mx(m)$  by solving  $m(x)$  for  $x(m)$  and inserting. The function  $\varphi(m)$  does contain the same information as  $y(x)$ , we can go back from  $\varphi(m)$  to  $y(x)$ .  $\varphi(m)$  is called the ‘Legendre transform’ of  $y(x)$ . Let us apply this to  $y=E$ ,  $x=S$  and  $m=\partial E/\partial S=T$ , and let’s call the resulting intercept ‘ $F$ ’.  $F$  will be a function of the ‘slope’  $T$  instead of  $S$ .

$$F(T, V, n, \dots) = E - TS.$$

We can also easily see the change of variables by using the chain rule on the differentials:

$$\begin{aligned} F(T, V, n, \dots) = E - TS &\Rightarrow dF = dE - TdS - SdT \\ dF = TdS - PdV + \mu dn + \dots - TdS - SdT \\ dF &= -SdT - PdV + \mu dn + \dots \end{aligned}$$

$F$  is now a function of  $T, V, \dots$  instead of  $S, V, \dots$ . It contains information equivalent to  $E$  because we can always transform back from  $F(T)$  to  $E(S)$ .  $F$  is called the Helmholtz free energy.

Similarly, if we want  $P$  as our variable instead of  $V$ , we can let  $y=E$ ,  $x=V$  and  $m=\partial E/\partial V=-P$ :

$$H(S, P, n, \dots) = E + PV \Rightarrow dH = TdS + VdP + \mu dn + \dots$$

$H$  is called the enthalpy. Just like the Helmholtz free energy, the enthalpy is also equivalent to  $E$ . We could even do it twice in a row, first for  $S$  and then for  $V$ , getting

$$G(T, P, n, \dots) = E + PV - TS = H - TS \Rightarrow dG = -SdT + VdP + \mu dn + \dots$$

$G=H-TS$  is called the Gibbs free energy. It is a function of  $T$  and  $P$ , but it also contains equivalent knowledge to  $E(S, V)$  that can be traced back to  $W$ . Now  $-S = \frac{\partial G}{\partial T}$  and  $V = \frac{\partial G}{\partial P}$  instead of the other way around. Using the explicit formula for the energy from c. on page 12, and applying it to a system with several components of mole number  $n_i$ , we can see that  $G$  has a particularly simple formula:

$$\begin{aligned} E = TS - PV + \sum \mu_i n_i \text{ and } G = E + PV - TS \\ \Rightarrow G = \sum \mu_i n_i \end{aligned}$$

The Gibbs free energy is just the sum over the chemical potentials of the individual chemicals “ $i$ ”,  $\mu_i=(\partial E/\partial n_i)_{S,V,\dots}$ . We will find the Gibbs free energy very useful in Chapter 5 when we want to know the equilibrium concentrations of reactants and products in a chemical reaction at constant  $T$  and  $P$ .

We’ll also find the chemical potential very useful. Based on the above equation, evidently we also have  $\mu_i=(\partial G/\partial n_i)_{T,P}$ . And since  $(\partial G/\partial T) = -S$  and  $(\partial S/\partial n_i) = s_i$  is the change of entropy per mole of substance “ $i$ ”,  $(\partial \mu_i/\partial T) = -(\partial^2 G/\partial T \partial n_i) = -s_i$  and similarly  $(\partial \mu_i/\partial P) = v_i$  and so forth. We call  $s_i$  the ‘molar entropy’ of substance “ $i$ ,”  $v_i$  its ‘molar volume,’ and so on. For an ideal substance, the molar volume is just the volume of one mole of substance. But in non-ideal situations, this may not be true!

**Thought experiment:** When some ions, such as  $Mg^{++}$  are dissolved in water, the volume of the resulting solution actually shrinks! That would mean (Le Châtelier’s principle) that the applying pressure should lower the chemical potential of  $Mg^{++}$  even more, so more will dissolve:  $(\partial \mu_i/\partial P) = v_i < 0$ . So the molar volume of  $Mg^{++}$  in water is actually negative. What could be the reason?

### 3. Heat flow and heat capacity

The Helmholtz free energy  $F(T, V, n)$  and the Gibbs free energy  $G(T, P, n)$  are already explicit functions of temperature.  $E$ ,  $S$  and  $H$  are not, but they could still change with temperature. For example,  $E$  is an explicit function of  $S$ ,  $V$ , and  $n$  (Postulate 1), and so  $dE = TdS - PdV + \mu dn$ . Consider the special

case where  $dV=0$  (no volume change, so no mechanical work can be done) and  $dn=0$  (no change in particle number, so no chemical work can be done). In that case,  $dE$  simplifies to  $dE = TdS$ . Also in that case, the only way to change the system's energy is by adding or removing heat, so  $dE = dq_{V,n}$ , where the subscripts indicate constant  $V$  and  $n$ . We can combine these two formulas as

$$dE = TdS = dq_{V,n} \text{ or } dS = \frac{dq_{V,n}}{T} \text{ (at constant } V \text{ and } n).$$

This tells us something interesting: if we get close to absolute zero temperature, even the tiniest amount of heat removed causes the entropy to drop by a lot. But that means it needs to increase by a lot somewhere else according to the second law. This is why it's really hard to cool things down close to absolute zero.

**Homework problem S3.4:** You already learned about the enthalpy  $H(S,P)=E+PV$ . Show that enthalpy is the heat flow when pressure is constant ( $dP=0$  and  $dn=0$ ), just like energy is the heat flow when volume is constant. To do so, use  $dH = TdS + VdP + \mu dn$  to write down a formula like the above one for  $S$ ,  $dq_{P,n}$  and  $T$  when pressure and mole number is constant. Look familiar compared to the equation above?

If the entropy increases a little bit due to a small temperature increase, the energy is also going to increase. According to the above formula,

$$\left(\frac{\partial E}{\partial T}\right)_{V,n} = T \left(\frac{\partial S}{\partial T}\right)_{V,n} \equiv C_V(T)$$

where the second equality ' $\equiv$ ' defines the heat capacity  $C_V$  at constant volume. The heat capacity simply tells us that the energy increases by a small amount if the temperature increases by a small amount (and vice-versa of course). We can rewrite the above formula as

$$dE = C_V(T)dT \text{ and } dS = \frac{C_V(T)dT}{T} \text{ (at constant } V \text{ and } n).$$

This allows us to calculate the change of energy and entropy of a system as a function of temperature if we now the heat capacity, even though  $E$  and  $S$  are not explicitly functions of  $T$ . Just like you did in Homework S3.4, we also can derive the analogous formulas at constant pressure,

$$dH = C_P(T)dT \text{ and } dS = \frac{C_P(T)dT}{T} \text{ (at constant } P \text{ and } n).$$

**Homework problem S3.5:** If  $S=S^{(0)}$  at the standard reference temperature  $T^{(0)}$  (e.g. 298 K) and constant pressure, show that at some other temperature,  $S=S^{(0)} + \int dT C_P(T)/T$  where the integration limits are from  $T^{(0)}$  to the other temperature. Derive a similar formula for  $H$ . We'll see later that this is useful for studying protein denaturation because disorder (entropy) of a protein increases as it unfolds at high temperature.

Finally, since  $C_V = T (\partial S/\partial T) = T/(\partial T/\partial S)$  and since  $T=\partial E/\partial S$ , we can eliminate  $T$  to get  $C_V = (\partial E/\partial S)/(\partial^2 E/\partial S^2)$ . So heat capacity tells us about the curvature (2<sup>nd</sup> derivative) of an energy vs. entropy plot. Via  $W(E) \rightarrow S(E) \rightarrow E(S)$ , heat capacity is directly tracable to Postulate 2.

**Homework problem S3.6:** By using  $H$  so we have a function of pressure, play the same game with the second derivative  $(\partial^2 H/\partial S^2)_{P,n}$  to derive the analogous formula for  $C_P$ .

#### 4. Calculation of thermodynamic "laws" from $W$ , $S$ , or $E$

To derive "laws" about a specific system, we need two ingredients: (1) Specify the basic properties of the system, and (2) use equations derived from Postulates 1 and 2.

Example: Fundamental relation for the “particles randomly moving in a large box” model:

(1) As we saw for identical particles of size  $V_0$  moving randomly in a box of size  $V$ ,

$$W = \frac{M!}{(M - N)! N!} \approx \left(\frac{V/V_0}{N}\right)^N$$

when  $M=V/V_0 \gg N$ .

(2) From Postulates 1 and 2, we proved  $dE = TdS - PdV + \mu dn + \dots$  and solving for  $S$ ,

$$dS = \frac{1}{T}dE + \frac{P}{T}dV - \frac{\mu}{T}dn + \dots,$$

Now let’s see what law this system obeys. From (1), we can calculate the entropy as in homework problem 2.3 to get

$$\begin{aligned} S &= k_B \ln W = -Nk_B \ln(V_0) + Nk_B \ln\left(\frac{V}{N}\right) \\ &= ns_0 + nR \ln\left(\frac{V}{n}\right) = ns_0 - nR \ln(c), \end{aligned}$$

where in the second line we used  $Nk_B = nAk_B = nR$ ,  $R = Ak_B \approx 8.31 \text{ J/mole/K}$  being the ‘gas constant’, which is just the Boltzmann constant multiplied by Avogadro’s number  $A = 6.02214076 \cdot 10^{23}$ . The concentration ‘ $c$ ’ is defined as  $c = n/V$ .  $s_0 = -R \ln(AV_0)$  is a constant that depends only on the properties of particles ( $V_0$ ) and fundamental constants.

Thus the entropy of a randomly moving gas (and also of an ideal solution, where solutes do not interact with one another) depends on the logarithm of the volume of the box, or concentration of the particles.

**Homework problem S3.7:** In the above formula,  $S_0 = ns_0$  in the last line lumps together all the constants. Derive  $S_0$  explicitly. Could  $S_0$  become negative? Should it be able to if we worked with the exact equation?

**In-class exercise:** By taking the derivative  $(\partial S/\partial V)$  in the above formula from (1), and the derivative  $(\partial S/\partial V)$  from the Postulate formula in (2), and equating the two, derive the “law” for particles randomly moving in a box much larger than the volume of the particles.

Now you see the power of statistical mechanics: We specify a system, ‘particles randomly hopping around in a box;’ and we assume energy is conserved (Postulate I) and all microstates of the same energy have same probability (Postulate II). Those two things, (1) specifying our system, and (2) using the postulates, are sufficient to derive the ideal gas law - and any other law describing molecules. You can see now that the ideal gas law is much more general than the way it is sometimes derived - the assumption ‘particles randomly hopping in a box’ is all that’s required to have an ideal gas, never mind all that stuff about forces, bouncing into container walls, etc.

**Homework problem S3.8:** Mechanical work  $dw = PdV$  is the work done by the environment on a system. Show that  $w = T\Delta S$  when only the volume of an ideal gas, not its temperature, is changed. [Hint: integrate  $dw$  from  $V_1$  to  $V_2$  using the ideal gas law, and show that  $\Delta S = S_2 - S_1 = nR \ln(V_2/V_1)$ , then combine the two equations.]

In order for work to be done, entropy has to increase somewhere: you can’t get something for nothing, as the saying goes.

## Chapter 4: Working at constant temperature - the Canonical Ensemble

All calculations in statistical mechanics can be done in the ‘microcanonical ensemble’ that we used so far, where all copies of the system are in microstates of the same energy with well-defined volume and particle number (isolated system). We saw that with just two postulates and some calculus, relations like  $PV=nRT$  or  $dS=dq/T$  can be derived. In particular, we were able to define a new variable “temperature” as the ‘quantity that is equal when two systems, between which heat is allowed to flow, come to equilibrium.’ Similarly, we defined what pressure and chemical potential are.

In reality, a system is never isolated; there is always a larger environment around the system that can exchange heat, or work, or volume, or particles in some combination or other. We could always treat the systems + environment as one larger isolated system, and apply the microcanonical ensemble and Postulate 2 directly. This works just fine, but it can be very cumbersome if you have to treat the environment explicitly all the time. Often it is more practical to just treat the system explicitly, and leave out the environment as much as possible. That’s the philosophy behind this chapter.

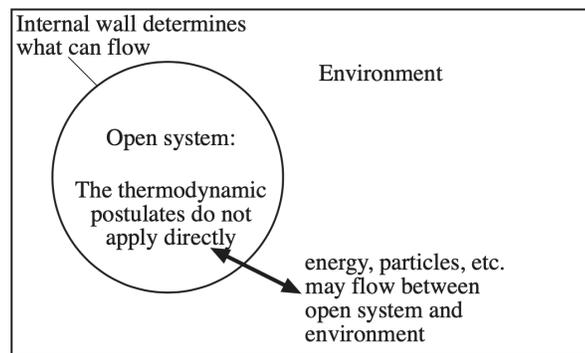


Figure: Open system inside of an environment. The two together form an isolated system to which the two Postulates apply, but not the open system by itself.

Consider heat flow as an example. Many reactions are done in open flasks, where heat can flow in and out. Thus at equilibrium, or if the reaction does not generate heat too rapidly (often the case in biochemistry) not the energy in the flask will be constant, but the temperature will be constant – the same temperature as the much larger lab environment, from/into which heat can flow to/from the flask, so thermal equilibrium is approximately maintained during the reaction.

Thus it would be nice to have formulas for systems at constant temperature. Such systems are called “canonical” in statistical mechanics, while systems at constant energy are called “microcanonical.” The ensemble of microstates of a system at constant temperature that belong to the same macrostate are called the “canonical ensemble,” in analogy to the “microcanonical ensemble” that we saw in Chapter 2.

To discuss a system at constant  $T$ , we need to embed the system inside a heat bath (the environment) and connect the system by a diathermal wall (that allows heat to flow) to the heat bath.

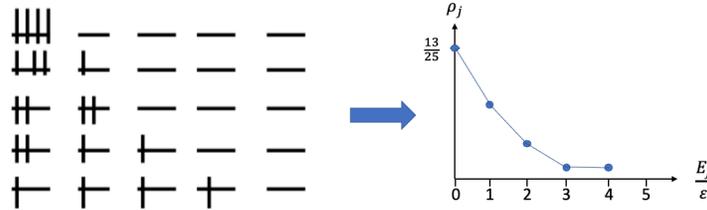
**Thought experiment:** Does this apply to single molecule experiments? A: Only the bath needs to be large so that  $W_{tot}$  and  $W_{bath}$  satisfy the large number approximations necessary for thermodynamics.  $W \equiv W_{sys}$  can be small. So the system could be a single molecule on a big surface, and if heat can flow from the molecule to the surface and *vice versa*, the molecule will still have the same temperature as the surface.

### 1. Deriving the canonical partition function from Postulate 2

Remember the definition of a partition function in Chapter 2: it is the number of microstates accessible when the system is in a given macrostate. For example, in the microcanonical ensemble,  $W(E)$  is the

number of microstates at constant energy  $E$ . What about the number of microstates in a macrostate at constant temperature  $T$ , instead of constant  $E$ ? That would be called the “canonical partition function” or “ $Z$ ” because the ensemble at constant temperature is the “canonical ensemble.”

**Homework problem S4.1:** What is the microcanonical partition function of an electron spin, when there is no magnetic field applied? (spin up and spin down have the same energy)?



**In class exercise:** Before deriving  $Z$  rigorously, let’s look at a simple model: In the above figure, consider  $N=5$  identical molecules, represented as horizontal lines, and  $k=4$  energy packets with energy  $\epsilon$ . If we assume these 4 packets are completely randomly distributed over the molecules, all possible cases are shown above: all 4 packets in one molecule, 3 packets in one molecule and 1 packet in another, etc. If a molecule is our open system, clearly it can have different values of the energy  $E_j$ , ranging from  $0\epsilon$  to  $4\epsilon$ . The probability of having one packet is  $1/25$  in the diagram on the left, the probability of having zero packets is  $13/25$ , and so on. So we can plot  $\rho_j$  as a function of  $E_j$ , shown on the right. It decreases roughly by a constant fraction for every increase of  $E_j$  by  $\epsilon$ . In fact, one can show that as  $N$  and  $k$  get very large,  $\rho_j$  decreases exactly by a constant fraction for every increase  $\epsilon$ , and this is the definition of a decaying exponential function:  $e^{-\beta(x+a)} = e^{-\beta x} \cdot e^{-\beta a}$ . Thus  $\rho_j \sim e^{-\beta E_j}$ . The probabilities  $\rho_j$  need to be normalized so they add up to 1, and so if we calculate the normalization factor  $Z = \sum_j e^{-\beta E_j}$ , then  $\rho_j = e^{-\beta E_j} / Z$ . Just as in postulate 1, the  $\rho_j$  are inversely proportional to the partition function  $Z$ , but they no longer have equal weight: microstates states at higher energy  $E_j$  are less likely than states at lower energy due to the “Boltzmann factor”  $e^{-\beta E_j}$ .

Now let’s do this rigorously. Let the system and environment together have constant energy  $E_{tot}$  (overall isolated system+environment). Then the number of ways the system can be at energy  $E_j$  is  $W(E_j) \cdot W_{env}(E_{tot} - E_j)$ . The number of ways the system can be at any energy is simply  $W_{tot}(E_{tot})$ . From this follows for the probability  $\rho$  of the system of being at energy  $E_j$  that

$$\rho_j^{(eq)} = \frac{W(E_j) \cdot W_{env}(E_{tot} - E_j)}{W_{tot}(E_{tot})}; \text{ using } S = k_B \ln W \implies$$

$$\rho_j^{(eq)} = W(E_j) e^{\{S_{env}(E_{tot} - E_j) - S_{tot}(E_{tot})\} / k_B}$$

where we used the ‘log trick’ in the second line to replace the multiplicative microcanonical partition function by the additive entropy.

We can simplify the exponent as follows: Let  $E$  be the (as yet unknown) *average* energy of the canonical ensemble of systems (= the ensemble at constant temperature). Let  $T$  be the temperature of the environment (and the system because heat can flow). Calling the system’s entropy simply “ $S$ ” instead of “ $S_{sys}$ ” for simplicity,

(a)  $\implies S_{tot}(E_{tot}) = S(E) + S_{env}(E_{tot} - E)$  (true at energy  $E$  or any other energy)

Taylor-expanding  $S_{env}(E_{tot} - E_j)$  around the average energy  $E$  of the system, i.e.  $\mathcal{Y}(x) = \mathcal{Y}(x_0) + \partial \mathcal{Y} / \partial x|_{x_0} \cdot (x - x_0) + \dots$ , where  $y = S_{env}$ ,  $x = E_j$  and  $x_0 = E$ ,

(b)  $\implies S_{env}(E_{tot} - E_j) = S_{env}(E_{tot} - E) + \left( \frac{\partial S_{env}}{\partial E_j} \right)_{E_j=E} (E_j - E) + \dots$

$$\begin{aligned}
&= S_{env}(E_{tot} - E) + \left(\frac{\partial S_{env}}{\partial E_{env}}\right) \left(\frac{\partial E_{env}}{\partial E_j}\right) (E_j - E) + \dots \\
&= S_{env}(E_{tot} - E) - \frac{1}{T} (E_j - E) + \dots
\end{aligned}$$

In the last step, we used  $1/T = \partial S/\partial E$  and  $dE_{env} = -dE_{sys}$  (first law of thermodynamics = energy conservation) to simplify the second term.

Inserting both (a) and (b) into the formula for  $\rho_j^{(eq)}$ ,

$$\rho_j^{(eq)} = W(E_j) e^{\{-\frac{1}{T}E_j + \frac{1}{T}E - S\}/k_B} = W(E_j) e^{\frac{1}{k_B T} \{-E_j + E - TS\}} = W(E_j) e^{\frac{1}{k_B T} \{-E_j + F\}}$$

$$\text{and finally } \rho_j^{(eq)} = W_j \frac{e^{-\beta E_j}}{Z}$$

In the second line, we define  $\beta=1/k_B T$  and  $W_j = W(E_j)$  as short hand notations, and  $Z = e^{-\beta F}$  is the normalization factor.

So what is  $\rho_j^{(eq)}$  in words? It is the probability of the system being in state  $j$  with energy  $E_j$  when the temperature is held constant at  $T$ . We call  $e^{-\beta E_j}$  the ‘Boltzmann factor.’ It says that the higher the energy  $E_j$ , the less likely the system will be at that energy; but as  $T$  increases, it becomes more likely that the system is at higher energy.

Again the total probability has to add up to 1, and we can use this to derive a formula for  $Z$ :

$$\sum_j \rho_j = 1 \Rightarrow \frac{1}{Z} \sum_j W_j e^{-E_j/k_B T} = 1 \text{ or } Z = \sum_j W_j e^{-E_j/k_B T} .$$

This is the result we got with our “energy packets in molecules” example. The only difference is that there we summed over every microstate  $j$  individually, and here we lump the  $W_j$  microstates with the same energy  $E_j$  together into a single term, since adding together  $e^{-\beta E_j} W_j$  times in a row is the same as just adding one term  $W_j e^{-\beta E_j}$  to the sum.

**The meaning of the canonical partition function.**  $Z$  is called the canonical partition function, in analogy to the microcanonical partition function  $W$ . Just like  $W$  counts the number of microstates accessible to an isolated system at constant energy,  $Z$  counts the number of microstates accessible to a system at constant temperature. For example, if  $T=0$ , then  $Z=W_0$ , the microcanonical partition function or “degeneracy” of the lowest energy level. This degeneracy is usually equal to 1: usually there is only one microstate without extractable energy (3<sup>d</sup> law of thermodynamics in Chapter 2). Thus  $Z=1$  at very low temperature because only the single lowest energy microstate can be populated. As the temperature increases,  $Z$  increases because more microstates can be populated. When  $k_B T \gg E_j$ , a state at energy  $E_j$  makes a full contribution

$$\lim_{x=E_j/k_B T \rightarrow 0} e^x = 1$$

to  $Z$ . If a state has  $E_j \gg k_B T$ , then it makes a small contribution to  $Z$ . Thus  $Z$  counts the number of accessible microstates, including the ones that are partially accessible.

The two different formulas we derived for  $Z$ ,

$$Z = \sum_j W_j e^{-\beta E_j} = e^{-\beta F} ,$$

have an important consequence. This equality connects the microscopic world of individual energy levels  $E_j$  and their degeneracies  $W_j$  (degeneracy = microcanonical partition function) with the macroscopic world of thermodynamic potentials, in this case  $F(T,V)$ , which is the potential useful at constant temperature, called the “Helmholtz free energy.” Once the microcanonical partition functions  $W(E_j)=W_j$  are known for each energy  $E_j$  by solving the Schrödinger equation  $H\Psi_j=E_j\Psi_j$ , then  $Z$  can be calculated. Once we know  $Z$ ,  $F$  can be calculated, and from  $F$  all other thermodynamic quantities can be calculated, since  $F$  is equivalent to  $W$ ,  $S$ ,  $E$ ,  $H$ , and  $G$ , as we saw in Chapter 3.

**Note:** If  $E$  is continuous, we replace the sum by an integration  $Z = \int_0^{\infty} dE W(E)e^{-E/k_B T}$ , where  $W(E)$  is the continuous density of states.

## 2. Computation of thermodynamic quantities from $Z$

We saw in the last chapter how to prove useful thermodynamic formulas like  $dS=dq/T$ , or  $PV=nRT$ . We can use the same method to calculate any average quantity we want from  $Z$ .

### *In-class exercise: What is the heat capacity of an RNA hairpin?*

Consider the following simple model for the reaction that interconverts a folded RNA hairpin to an unfolded RNA hairpin. The figure below shows a simple 2-D lattice model for an RNA with four bases, two of which can base pair in the “stem.” In this model, we can count all the states that are non-superimposable by 2-D rotation:

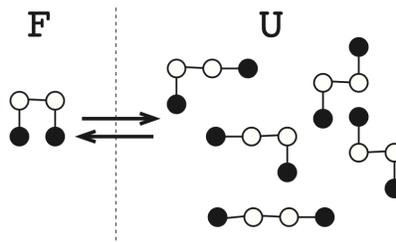


Figure: Folding/unfolding reaction of an RNA hairpin in a simple 2-D lattice model that allows only  $90^\circ$  hinge motions of the bases, and counts only one interaction energy (base pairing energy), between the two black bases.

Let the base pairing energy between the two black bases be  $-\varepsilon$  kJ/mole. Renormalizing the minimum energy to 0, we have  $E_F = 0$   $W_F=1$  and  $E_U=+\varepsilon$  and  $W_U=5$ . The partition function for this system is

$$Z = W_F + W_U e^{-\varepsilon/RT}.$$

The probability of being in states F and U is

$$\rho_F = \frac{W_F}{Z}, \rho_U = \frac{W_U e^{-\varepsilon/RT}}{Z},$$

and the ratio of concentrations for the above reaction is therefore

$$K = \frac{[U]}{[F]} = \frac{\rho_U}{\rho_F}.$$

We will prove in chapter 5 that this ratio, known as the equilibrium constant  $K$ , can be calculated directly from the Gibbs free energy introduced in chapter 3.

The average energy of the system is given by

$$U = E_F \rho_F + E_U \rho_U = \frac{W_F}{Z} 0 + \frac{W_U e^{-\varepsilon/RT}}{Z} \varepsilon = \frac{W_U \varepsilon e^{-\varepsilon/RT}}{W_F + W_U e^{-\varepsilon/RT}}.$$

We now have the energy as a function of temperature and the parameters  $W_F$ ,  $W_U$  and  $\epsilon$ . Now take the derivative of the energy as in with respect to temperature to obtain

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = \left(\frac{1}{R}\right) \left(\frac{\epsilon}{T}\right)^2 \rho_U \rho_F$$

This heat capacity has a maximum at  $\rho_U = \rho_F = 1/2$ , in the middle of the folding/unfolding transition. Thus the ‘melting temperature’ of an RNA piece could be determined by calorimetry by looking for the heat capacity maximum.

Let’s calculate this melting temperature  $T_m$ , where  $\rho_U = \rho_F = 1/2$ : according to the definition of  $\rho_F$  above, at the melting temperature  $1/2 = 1/Z$  (since  $W_F = 1$ ) or  $2 = 1 + W_U \exp(-\epsilon/RT_m)$ , or  $T_m = \epsilon / (R \ln(W_U))$ . This should make sense: the higher the interaction energy  $\epsilon$ , the harder it will be to melt the RNA; the more unfolded microstates are available, the easier it will be to melt the RNA.

Note that a simple model like the above will not be perfectly accurate. For example, by making it 2-D, we are grossly underestimating the number of unfolded configurations compared to the folded ones. We could easily remedy this by writing  $\rho_U$  as

$$\rho_U = 1 - \rho_F = \frac{W_U/W_F e^{-\epsilon/RT}}{1 + W_U/W_F e^{-\epsilon/RT}}$$

and making the ratio  $W_U/W_F$  an adjustable parameter in our fitting model. Or we could do a better calculation by, say, running molecular dynamics to count configurations and evaluate energies. The important thing is that statistical mechanics lets you derive realistic model functions for your experiment, instead of fitting the data by a polynomial or some other function that has no physical meaning and provides no physical insight. In the above equation for  $C_V$ , the physical insight is that the heat capacity is directly related to the base pairing energy  $\epsilon$ , depends on the ratio  $(\epsilon/T)^2$ , has a peak, and depends on the ratio  $W_U/W_F$  (which is probably much larger than 5 in reality), and that the RNA melts at a temperature  $T_m = \epsilon / (R \ln(W_U))$ . That’s a lot of useful things you have already learned about RNA folding just by thinking about a simple model before even touching the calorimeter to do the experiment.

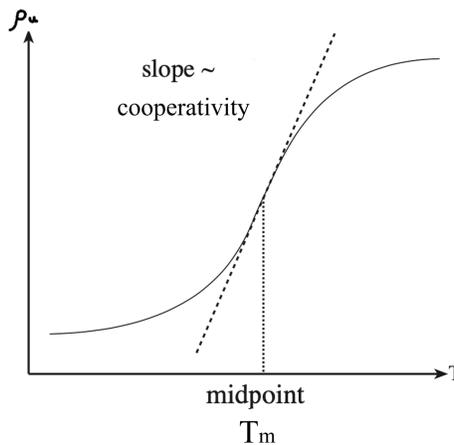


Figure: Plot of the unfolded state fraction  $\rho_U$  as a function of temperature. The curve is signoidal or ‘cooperative’ with a maximum slope shown by the dotted line. We say the RNA hairpin ‘unfolds at temperature  $T_m$ ’, that is, unfolding is defined as the midpoint of the transition.

**Homework problem S4.2:** Derive a simple formula for the average energy of a system at constant temperature. Start with  $Z = e^{-\beta F}$ , write out  $F$  in terms of  $E$  and  $S$  in the exponent, then take the derivative of  $Z$  with respect to  $\beta$  and solve for  $E$ .

**In-class exercise:** What is the energy and heat capacity for  $n$  moles of ideal (= non-interacting) gas particles?. A: To do this, let's (1) realize that a quantum particle-in-a-box with energy  $E = \hbar^2 \{n_x^2 + n_y^2 + n_z^2\} / (8mL^2)$  from the quantum part of the course is an ideal gas particle in a box of volume  $V = L^3$ . (2) Write  $Z$  by summing over all possible values of the quantum numbers  $n_x$ ,  $n_y$ , and  $n_z$  from 0 to  $\infty$ . (3) Since the energy level spacing goes to 0 as  $L$  gets large (macroscopic box), we can approximate the summation by an integral and get a simple formula for  $Z_{1D} = (2\pi mk_B T / \hbar^2)^{1/2} L$ . For a particle in 3-D, this then becomes  $Z = Z_{1D}^3$ . (4) Use the formula for getting  $E$  from  $Z$  that you are deriving on the homework to get  $E(T)$ . Finally, use the definition of heat capacity  $C_v$  from Chapter 3 to calculate the heat capacity. Do these formulas look familiar?

From the above, we conclude for a gas that the highest randomness (equilibrium) is achieved when energy is distributed completely randomly among particles, so on average each 'degree of freedom' that can store energy has  $1/2 k_B T$  of energy. That's why a monatomic gas particle has energy  $3/2 k_B T$ , a mole of such particles has energy  $3/2 nRT$ , and a spring, which can store energy as potential and kinetic energy has energy  $2/2 k_B T = k_B T$ . Although we proved it here just for an ideal gas, this principle can be proven valid at high temperatures for any system where we can replace the quantum sums by integrals, and treat the energy levels as a 'continuum'. It is perhaps one of the most important principles in science, useful in discussing anything from Maxwell's demons (see below) to global warming (see next chapter):

**Equipartition principle: in a classical system at temperature  $T$ , every degree of freedom on averages stores an energy  $1/2 k_B T$ .**

**Homework problem S4.3: Derive a formula for average pressure in a system at constant temperature that directly relates  $P$  to  $Z$ .** Hint: remember from chapter 3 that  $P = -(\partial E / \partial V)$ . The analogous formula is still true for the Helmholtz potential  $F$ , and  $F$  is related to  $Z$  as we proved earlier in this chapter.

**Homework problem S4.4: Derive a formula for the entropy of a system at constant temperature in terms of  $Z$ .** Hint: Once we have  $E$  and  $F$ , it is easy to obtain  $S$ !

Remember that  $F$  is a function of  $T$  and  $S$  is a function of  $E$ , so to get a proper fundamental relation, you have to substitute  $T = \partial E / \partial S$  as a function of  $E$  and/or  $S$  everywhere. Thermodynamics provides us with the manipulations to get any variable we want once we have computed the partition function.  $Z$  (or  $W$ ) contain all knowledge about the equilibrium properties of a system, just like  $E(S)$  and  $F(T)$  do.

**In-class exercise: A diatomic molecule with spring constant  $k$  is heated to a temperature  $T$ . What is the average (root-mean-squared) length  $L$  by which the molecule is compressed or expanded because of heating?**

**Answer:** The applied force that stretches the spring is  $f = kL$ , where  $L$  is the displacement from the equilibrium bond distance. Instead of  $E = TS - PV$  for a 3-D system, we have  $E = TS - fL$  for a 1-D spring. Note that  $fL = kL^2$ , not  $1/2 kL^2$ .  $E$  is not the potential energy, but the total energy, and when the spring is heated, half the energy is stored in potential energy, the other half in kinetic energy (the classical equipartition principle). I use lower case  $f$  for the force here so we don't confuse it with free energy  $F$ .

From our study of quantum mechanics, the energy of the spring is  $E_n = \hbar \omega n$ , if we shift to the zero point energy as our reference energy. Now we can calculate  $Z$

$$Z = \sum_{n=0}^{\infty} 1 \cdot e^{-\frac{\hbar\omega n}{k_B T}} = \sum_{n=0}^{\infty} e^{-\left(\frac{\hbar\omega}{k_B T}\right)^n} = \frac{1}{1 - e^{-\frac{\hbar\omega}{k_B T}}}$$

$$\Rightarrow F = -k_B T \ln Z = +k_B T \ln \left(1 - e^{-\frac{\hbar\omega}{k_B T}}\right)$$

And since  $E = TS - fL$  for a spring

$$\Rightarrow F = E - TS = -fL; f = kL \Rightarrow F = -kL^2 \Rightarrow L^2 = -\frac{F}{k} = +\frac{k_B T}{k} \ln Z$$

$E = TS - fL$  for a spring, where “ $-fL$ ” replaces “ $-PV$ ” for a gas because the spring is one-dimensional. So  $F = E - TS = -fL$ , and since  $f = kL$  we can write  $F$  as  $F = -kL^2$ . Solving for  $L^2$ , we get  $L^2 = -F/k = (k_B T/k) \ln Z$ . Therefore we have for the root-mean-square displacement of the spring

$$L_{rms} = \sqrt{\frac{k_B T}{k} \ln Z}.$$

This remarkable result tells us how the root-mean-square thermal fluctuations of a diatomic molecule depend on temperature  $T$ , spring constant  $k$  and number of states populated  $Z$ . For a macroscopic spring,  $L_{rms}$  would be rather small compared to the length of a spring, but for a molecule-sized spring, it could be quite large relative to the length of the spring. This is the reason we cannot build a [Maxwell Daemon](#) by making a little door between two chambers held tight by a spring on one side so “particles can only go through one way.” In reality, the random thermal motion of all the particles would also make the spring vibrate so much that particles could get through either way (or if  $k$  is too large, neither way). Equipartition of energy between particles and springs messes up the daemon. There’s just no escaping the Second Law!

We’ve come a long way. Using just two assumptions about Nature, (1) Newton’s equations, or if needed for accuracy, the Schrödinger equation, are true, and (2) states of equal energy have equal probability, we are able to derive at what temperature an RNA molecule unfolds, or how much the bond length of a molecule vibrates at temperature  $T$ , or that a gas has energy  $^{3/2}nRT$ . In the next Chapter, we’ll continue our brief tour of statistical mechanics and thermodynamics by looking closely at the Gibbs free energy, and what it tells us about equilibria of chemical reactions at constant temperature and pressure. Of course, you already know the answer from freshman chemistry: we can calculate an ‘equilibrium constant’ that tells us how much of each chemical we’ll get at the end of the reaction.

## Chapter 5. Chemical equilibrium and the free energy.

### 1. Rewriting the second law for an open system (reaction)

In chapter 2 we derived the second law of thermodynamics from statistical mechanics:

$$S_{tot}(t>0) > S_{tot}(t=0),$$

or

$$dS_{tot} > 0$$

for a spontaneous process in an isolated system (constant  $E$ ,  $V$ , ...) with entropy  $S_{tot}$ . But in many chemical reactions, the system is open, and  $T$  and  $P$  are constant, not  $E$  and  $V$ .  $T$  and  $P$  are constant because the system (like a beaker in the figure on the next page) is connected to large environment (like a lab) kept at constant  $T$  and  $P$ ; so when equilibrium has been reached,  $T$  and  $P$  in the beaker are the same as in the lab, and this is true even during the reaction, if the reaction runs slowly enough.

Can we find a specific version of the second law for the case of an open system, when energy can flow between system and environment?

As an example, consider a solar photon as our open system, and the Earth as its environment. Photons actually undergo very simple reactions on Earth and in the atmosphere by being ‘absorbed’ (used up) and ‘emitted’ (newly made).

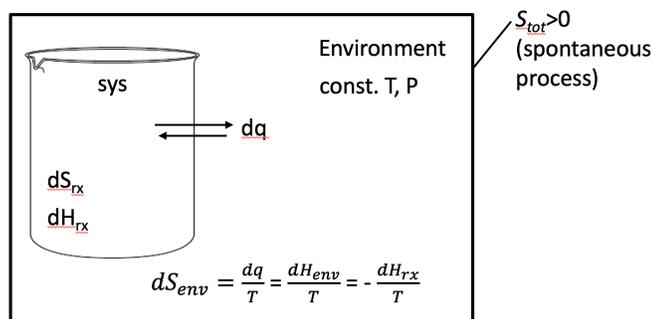
**Homework problem S5.1:** When a visible solar photon of energy  $h\nu \sim k_B T_{solar}$  ( $T_{solar} \sim 6000$  K) hits the Earth’s surface, it is absorbed, but the energy does not disappear. Instead, it is either re-emitted as several ‘thermal’ photons with energy  $\sim k_B T$  ( $T \sim 300$  K), or transferred as heat energy, also in packets  $\sim k_B T$  by the equipartition principle. Let’s assume for a moment the photon just splits into equal lower energy  $\sim k_B T$  thermal energy photons. The reaction here is  $\nu_{solar} \rightarrow N \nu_{thermal}$ . How many  $N$  ‘thermal’ photons can be made from a solar photon by energy conservation?

**Homework problem S5.2:** Assume that the photons behave like an ideal gas in a box, which we’ll take to be unit volume  $V=1$ . Use the formula for  $Z$  of a gas from the last chapter, Planck’s law  $E=h\nu$ , and Einstein’s formula  $E/c^2 = m$  to write down the partition function for a photon per unit volume in terms of only  $k_B$ ,  $T$ ,  $c$ ,  $\nu$ , and  $h$ . [Note: your formula is a ballpark estimate because photons really are massless and really  $E/c^2 = (m^2 + (p/c)^2)^{1/2}$ ; but it will be very close to the right answer below.]

**In class problem:** Let’s say 20 infrared photons are made out of one solar photon, and remember that  $Z_{tot} = Z_1 Z_2 Z_3 \dots Z_N / N!$  for  $N$  indistinguishable particles. (Entropy adds, partition function multiplies, indistinguishable particles reduces  $Z$  by a factor  $N!$ ). With the partition function per unit volume of one thermal photon being  $8\pi(k_B T / ch)^3$ , how much does the number of microstates go up when a solar photon is converted to infrared photons? Is it better to do that calculation with entropy, which is additive? A: compare  $Z_i^N / N!$  at 300 K and  $N=20$  to  $Z$  for the original photon at 6000 K.  $S$  would do this with more manageable-sized numbers.

CO<sub>2</sub> molecules absorb strongly in the infrared, whereas O<sub>2</sub> and N<sub>2</sub> do not. So thermal photons re-emitted from the Earth’s surface towards space can be absorbed by CO<sub>2</sub> in the atmosphere and re-emitted in a random direction, warming the Earth’s surface rather than letting the photon escape.

**Thought experiment:** If a photon is re-emitted in a random direction in the atmosphere near the Earth’s surface, what is the probability it will reach space?



Going back to our question: Can we find a version of the second law specific for the case of an open system at constant temperature  $T$ , instead of constant energy  $E$ ? The answer is yes. As shown in the above figure, we will consider an isolated system ‘tot’ that satisfies the second law, and contains a large environment ‘env’ at constant temperature and a reaction ‘rx,’ which is our open system coupled to the environment by heat flow. Thus  $T$  in the reaction and environment is the same, and as long as the reaction runs slowly enough,  $T$  will remain the same even if the reaction releases heat because the environment is large compared to the system.

First, let’s make use of the additivity of the entropy and split the entropy of the isolated system into the reaction and environment parts. The change in entropy is given by

$$dS_{tot} = dS_{rx} + dS_{env} > 0.$$

Second, let’s get rid of  $dS_{env}$ , a quantity that is hard to measure. We derived in chapter 2 that  $dS=dq/T$  is the change of entropy due to heat flow at either constant pressure (when  $dq=dH$ ) or constant volume (when  $dq = dE$ ). The figure above illustrates the case of constant  $P$  in addition to constant  $T$ , the most common case in experimental labs. Thus,

$$dS_{tot} = dS_{rx} + dH_{env}/T > 0,$$

with a similar formula replacing  $H$  by  $E$  if volume is constant instead of pressure.

Third, the critical step: by using energy conservation, we can get rid of  $dH_{env}$  in favor of  $dH_{sys}$  because  $dH_{env}=-dH_{sys}$ . Heat that flows out of the system must flow into the environment, or vice versa. We now have

$$dS_{tot} = dS_{rx} - dH_{rx}/T > 0$$

This is a remarkable result. We have expressed the total entropy change that obeys the second law, but is difficult to measure, by quantities just related to the system, which are much easier to measure and tabulate for future use. At constant volume, a similar formula replacing  $dH_{rx}$  by  $dE_{rx}$  is obtained.

It is important to note that we cannot substitute  $dS_{rx}$  by  $dq_{rx}/T$ . The entropy change in the reaction has many sources, not just the heat flow, whereas the entropy change in the environment, as far as its interaction with the system is concerned, has only one source: heat flow from the reaction. For example, if the reaction consisted of mixing two inert gases, the entropy  $S_{rx}$  would increase ( $dS_{rx}>0$ ), but  $dq_{rx}=0$  in that case. Or if the reaction consisted of making a bond,  $dS_{rx}<0$  (two atoms move together afterwards), but  $dq_{rx}<0$  (making the bond releases energy, say in the form of a photon released into the environment since heat flow is allowed).

Finally, we can rewrite the above formula in terms of a quantity we already derived. Recall that the Gibbs free energy is  $G=H-TS$ . Thus its total differential is  $dG=dH-TdS-SdT$ , but at constant temperature, this simplifies to  $dG=dH-TdS$ . If we multiply the equation for  $dS_{tot}$  by  $-T$  on both sides and realize that this flips the sign of the inequality (e.g. if  $7>5$ , then  $-7<-5$ ), we have

$$-TdS_{tot} = dH_{rx} - TdS_{rx} = dG_{rx} < 0 \text{ at constant pressure.}$$

If volume is constant instead of pressure, we obtain the analogous equation

$$-TdS_{tot} = dE_{rx} - TdS_{rx} = dF_{rx} < 0 \text{ at constant volume.}$$

This is the second law of thermodynamics for an open system embedded in an environment that allows heat to flow. It expresses the total change of entropy, which must be positive according to the second law, in terms of *only* reaction quantities: reaction entropy and reaction heat released, or reaction Gibbs free energy or Helmholtz free energy. The  $-TdS_{rx}$  term accounts for the entropy change of the reaction, and the  $dE_{rx}$  or  $dH_{rx}$  term accounts for the entropy change of the environment due to heat flow to or from the reaction.

The above is the reason thermodynamics is such a useful theory, and can set bounds on quantities, such as global warming, using very general formulas. One just has to make sure that all the important reactions of one's system are included.

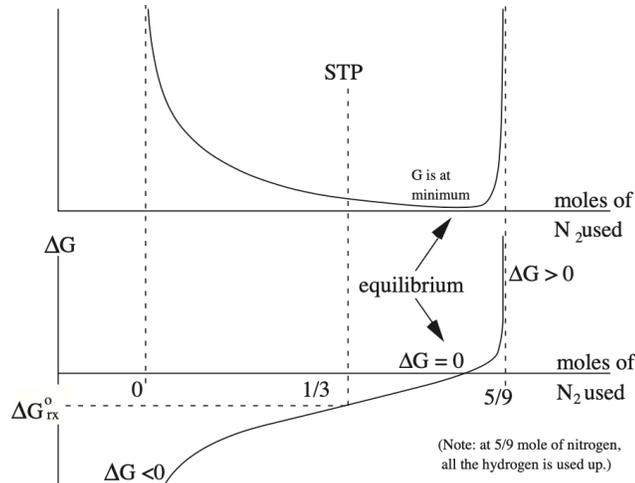


Figure: How the free energy of the system  $G$  (top), and the derivative of the free energy with respect to reaction progress,  $\partial G/\partial x$ , often called  $\Delta G$  (bottom) depends on the reaction progress “ $x$ ”. In this example, we are looking at the reaction  $N_2+3H_2\rightarrow 2NH_3$  and we have  $1.66 = 5/3$  moles of  $H_2$  at the start of the reaction.

If the reaction progresses by  $dx$  moles of a product when the free energy changes by  $dG_{rx}$ , we can write the formula  $dG_{rx} = dH_{rx} - TdS_{rx} < 0$  as

$$\left(\frac{\partial G_{rx}}{\partial x}\right)_{T,P} = \left(\frac{\partial H_{rx}}{\partial x}\right)_{T,P} - T\left(\frac{\partial S_{rx}}{\partial x}\right)_{T,P} < 0 \text{ or}$$

$$\Delta G = \Delta H - T\Delta S < 0$$

where the derivatives are written in short hand notation as  $\Delta G \equiv (\partial G/\partial x)$  etc, and have units of kJoules/mole. In the second equation and the above Figure, we dropped the subscript “rx” because it is understood from now on that we are talking about the reaction (open system), not the environment or the overall isolated system.

Since  $\Delta G = -T\Delta S_{tot}$ , it follows that for a reaction at constant temperature and pressure,

$$\Delta G(P,T) < 0 \text{ (spontaneous reaction)}$$

$$\Delta G(P,T) = 0 \text{ (equilibrium)}$$

$$\Delta G(P,T) > 0 \text{ (spontaneous reverse reaction)}$$

$\Delta G$  tells us how much the free energy  $G$  changes if the reaction proceeds by some infinitesimal amount “ $dx$ ”. Thus if  $G$  has units of Joules,  $\Delta G$  has units of Joules/mole. This is just another statement of the second law of thermodynamics, but for the special case of an open system (reaction) at constant temperature and pressure. Of course if our reaction does not satisfy these assumptions, we can always go back to the full statistical mechanical analysis to check if  $S(t>0) > S(t=0)$  to see if the reaction is spontaneous.

**Homework problem S5.3:** On the figure above, mark with an arrow a place where the reaction spontaneously makes more ammonia, and a place where it makes more nitrogen and hydrogen? What is the slope of  $G$  in each case, positive or negative? Are there more places in the plot where ammonia will be made, or hydrogen will be made? So if I pick random concentrations between 0 and 1 moles/l of hydrogen, nitrogen and ammonia as starting conditions, will the reaction most likely go forward or backward?

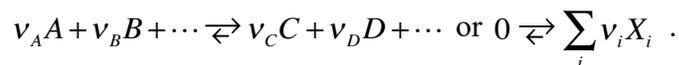
$\Delta G$  is made up of two pieces, an enthalpy change, and an entropy change: the more the enthalpy is lowered, the more negative the derivative;  $\partial G/\partial x$  and the more the entropy increases, the more negative the derivative. Thus systems tend to either go towards lower enthalpy (or energy at constant volume), or towards higher disorder, and these two effects always compete and balance. Going towards lower energy occurs because then heat is released into the environment, making it more disordered; going towards higher entropy, or more microstates  $W$ , occurs because the system then directly occupies more microstates, which is more likely.

## 2. Concentration dependence of the free energy $G$ in the ideal case

For a mole of ideal gas or an ideal solution (noninteracting particles that move around randomly in a box), we proved in Chapter 3 that  $s=s^{(0)}+R\ln(V)=s^{(0)}-R\ln(c)$  for a mole of substance (the ‘molar entropy’).

**Thought experiment:** If you have a solution filled with ions that can react individually, but they are far enough apart so they do not attract each other strongly and so heat flow can keep the temperature constant, will the energy released by the reaction of one ion affect another?

Also for noninteracting particles, the enthalpy does not depend on volume, so  $h=h^{(0)}$ , where  $h^{(0)}$  is the enthalpy for one mole of particles. Note that  $s^{(0)}$  and  $h^{(0)}$  could still depend on temperature, as discussed already in chapter 3. Consider the free energy for a mixture of different noninteracting particles “i” obeying the reaction



In the general notation for a reaction on the right,  $X=A,B,C,\dots$  and the stoichiometric coefficients  $\nu_i$  are positive for products, negative for reactants. Don’t mix up stoichiometric coefficients with frequency  $\nu$ , although books (and I!) use the same Greek letter.

If the reaction starts at mole numbers  $n_i^{(0)}$  and is allowed to advance by  $x$  moles,

$$\begin{aligned} G &= \sum_i n_i \mu_i \\ &= \sum_i (n_i^{(0)} + \nu_i x) \mu_i \end{aligned}$$

Taking the derivative with respect to  $x$ , to get the change of free energy as the reaction proceeds,

$$\left(\frac{\partial G}{\partial x}\right)_{T,P} = \Delta G = \sum_i \nu_i \mu_i$$

Now replacing  $\mu_i=h_i-Ts_i$  for each reactant/product and simplifying,

$$\begin{aligned} \Delta G &= \sum_i \nu_i h_i - T \sum_i \nu_i s_i \\ \Delta G &\approx \sum_i \nu_i h_i^{(0)} - T \sum_i \nu_i s_i^{(0)} - \nu_i R \ln(c_i), \end{aligned}$$

where we used the formula for how entropy depends on volume or concentration that we derived earlier for an ideal gas or solution. Thus,

$$\Delta G \approx \sum_i \nu_i \mu_i^{(0)} + T \sum_i \nu_i R \ln(c_i)$$

The approximation of the “ideal” case occurs between lines 1 and 2. In line 3, we just combined  $\mu_i^{(0)} = h_i^{(0)} - T s_i^{(0)}$ .

**Homework problem S5.4:** Since  $\mu_i = h_i - T s_i$ , insert the formula for  $s_i = s_i^{(0)} - R \ln(c)$  that we derived at the end of Chapter 3 to show directly that the chemical potential for chemical species “i” can be written as  $\mu_i = \mu_i^{(0)} + RT \ln(c)$ , and what is the constant  $\mu_i^{(0)}$ . Note that this strictly applies only for ideal solutions or gases.

If we name the first sum  $\Delta G^{(0)}$  (how much, in kJ/mole, the free energy of the reaction changes at the initial condition “(0)” when  $x$  changes by a small amount), and in the second term, we put the sum over  $\nu_i$  into the logarithm:

$$\Delta G = \Delta G^{(0)} + RT \ln(Q), \text{ where } Q = \prod_i c_i^{\nu_i}$$

Here,  $Q$  is the **mass action law** coefficient, and the above equation is the **mass action law**. It allows you to compute the free energy change if a reaction proceeds by a small amount given the concentrations of reactants and products, and their free energies at  $T$  and  $P$ . Values of  $\Delta G^{(0)}$  are usually tabulated under ‘standard conditions’ (temperature  $T = T^{(0)} = 298$  K,  $P = P^{(0)} = 1$  atm, and concentration  $c^{(0)} = 1$  molar).

The free energy has units of Joules. **Its derivative  $\Delta G$  has units of Joules/mole, and tells us how much the free energy changes per mole when the reaction proceeds by an infinitesimal amount.** At equilibrium,  $G$  has reached a minimum, and  $\partial G / \partial x = \Delta G = 0$ . If  $\Delta G$  were  $< 0$ , the reaction would go forward; if  $\Delta G$  were  $> 0$ , it would have ‘overshot’ equilibrium and go backward. Thus at equilibrium,

$$0 = \Delta G^{(0)} + RT \ln(Q_{eq}) \Rightarrow K_{eq} \equiv Q_{eq} = e^{-\frac{\Delta G^{(0)}}{RT}}.$$

Note that at equilibrium, we like to give the mass action coefficient a special name, “ $K_{eq}$ ”. We call it the “equilibrium constant.” Note that we have one last missing ingredient in our statistical mechanical theory of reaction equilibria:  $K_{eq}$  or equivalently  $\Delta G^{(0)}$  have not yet been derived from first principles.

### 3. Calculating the equilibrium constant from molecular properties by using statistical mechanics

We won’t do it here in detail, but you can probably imagine by now that we don’t have to measure the free energy or rate coefficient  $k$  experimentally, we can actually calculate them from statistical mechanics. The way you do it is that the chemical potential (like anything else!) can be calculated if you know the partition function of the molecules (reactants and products) participating in the reaction. At constant volume (so we can use the Helmholtz potential  $F$ ), as we saw in Chapter 4 (p. 13) that

$$\mu_i = \left( \frac{\partial F}{\partial n_i} \right)_{T,V} = A \left( \frac{\partial F}{\partial N_i} \right)_{T,V},$$

where  $A$  is Avogadro’s number.  $F$  can be related to  $Z$  by  $F = -k_B T \ln Z$ .  $Z$  in turn is given by the product of the partition functions  $z_i^{N_i} / N_i!$ . Thus,

$$\mu_i = -RT \frac{\partial}{\partial N_i} (N_i \ln z_i - N_i \ln N_i + N_i)$$

if we use Stirling’s approximation for  $N_i!$ . Finally,

$$\mu_i = -RT \ln \frac{z_i}{N_i}$$

And of course we saw above that

$$\Delta G^{(0)} = \sum v_i \mu_i^{(0)} \text{ and } K_{eq} = e^{-\Delta G^{(0)}/RT}.$$

So, if we know the energy levels and degeneracies of each of our molecules by solving its Schrödinger equation, we can calculate

$$z_i = \sum W_j e^{-\frac{E_j}{k_B T}}$$

for molecule “i”, then use  $z_i$ , to calculate  $\mu_i = \mu_i^{(0)}$  by picking standard conditions, then combine the  $\mu_i^{(0)}$  with the stoichiometric coefficients and voila - plug in to get  $K_{eq}$ . This is often more accurate than doing the experiment, although for some reactions, the quantum calculations are still not good enough. For a typical organic reaction, electronic structure calculations now provide an accuracy of about 1 kcal/mole  $\sim$  4 kJ/mole, the main limiting factor being approximations made to solve the many-electron Schrödinger equation accurately. We saw it's a bear without a computer when we were doing quantum mechanics!

**Homework problem S5.5:** Let's calculate the equilibrium constant for one example, the folding of the RNA from macrostate U to macrostate F in the reaction  $U \rightleftharpoons F$  in the previous chapter, so  $v_F = +1$  and  $v_U = -1$  for the stoichiometric coefficients going towards folded product. Using the partition function  $Z$  for the folded RNA in the previous Chapter as  $Z_F = W_F$ , and for the unfolded RNA as  $Z_U = W_U e^{-\epsilon/RT}$ , write down the formula for  $\Delta F$  and the equilibrium constant  $K_{eq}$ . These are two of the most important equations to describe at what temperature an RNA molecule or a protein folds. [Note: it's  $\Delta F$  instead of  $\Delta G$  because our canonical partition function formula from the last chapter is at constant temperature and volume, not temperature and pressure; but protein solutions are not very pressure sensitive, so the formula for  $\Delta G$  is almost the same.]

## Chapter 6. Out of equilibrium: Brownian motion and drift velocity

While we are interested in seeing where reactions will go in equilibrium, eventually that equilibrium is going to be perturbed, and we will go to a new equilibrium. It's just a matter of time.

**Thought experiment:** the pencil I just put on the table is in mechanical equilibrium. How likely do you think it will be in this equilibrium 10 seconds from now? How about tomorrow?

Clearly, perfect equilibrium is never reached in reality, although it may be reached long enough that we can do useful calculations like in the last Chapter. Eventually, though, things get moving. Specifically, how do molecules tend to move around? We can of course fling molecules around, so they follow a trajectory like Newton's law. But even if we don't touch them, we notice they actually move about. Why? Because of equipartition, at least classically there is energy  $k_B T/2$  in every degree of freedom, and molecules constantly kick each other randomly to maintain or evolve towards that average energy per active degree of freedom.

**1. Brownian motion** Let us consider the trajectory of a 'test' particle in a 'heat bath of solvent particles: for example, a protein molecule surrounded by water molecules that can bump into the protein randomly due to heat. On average the forces from the solvent particles cancel, but at any instant a few more may hit the test particle from one side than from the other side. The test particle is getting buffeted around by solvent particles.

At very short times, the motion is predictable by Newtonian mechanics  $F=ma$ :

$$x(\Delta t) = x(0) + v(0)\Delta t; \quad v(\Delta t) = \frac{F(0)}{m} \Delta t + v(0),$$

After many such events, the motion becomes unpredictable because the forces  $F_{random} = F(0), F(\Delta t), F(z\Delta t)...$  change randomly with each collision, see [BD Simulation](#) at:

[https://en.wikipedia.org/wiki/Brownian\\_motion](https://en.wikipedia.org/wiki/Brownian_motion)

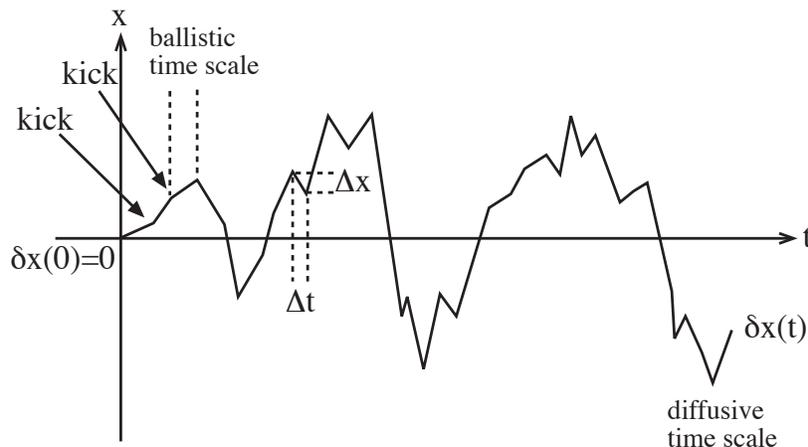


Figure: Random walk of a test particle in one dimension. At each time step  $\Delta t$  the particle receives a kick. After  $k \gg 1$  kicks, the position of the particle is the sum of many ( $=k$ ) random variables, thus it is a Gaussian random variable.

Each of the hops  $\Delta x$  that the particle makes during the mean collision time  $\Delta t$  is random. Thus, the displacement of the particle from the initial position  $x_0$  after  $k \gg 1$  hops at time  $t = k\Delta t$  is equal to  $\delta x(t) = x - x_0 = \sum_{j=1}^k \Delta x_j$ , and it is a Gaussian random variable by the central limit theorem

(C.L.T.). This is true NO MATTER WHAT the probability distribution of  $\Delta x$  looks like, so we need to know nothing about it. Thus, if

$$\delta x(t) = \sum_{j=1}^k \Delta x_j, \text{ where } k = \frac{t}{\Delta t} \xrightarrow{\text{C.L.T.}} \langle \delta x^2 \rangle = k \langle \Delta x^2 \rangle = \frac{\langle \Delta x^2 \rangle}{\Delta t} t$$

If we rename the constant  $\langle \Delta x^2 \rangle / \Delta t = 2D$ , we obtain

$$\langle \delta x^2 \rangle = 2Dt .$$

This is the law of Brownian motion, which states that the mean square displacement of a particle grows linearly with time, i.e. the root mean square displacement only grows as the square root of time.  $D$  is known as the diffusion coefficient; the larger  $D$  is, the more rapidly the particle moves away from the initial position. It has units of length<sup>2</sup>/time. Whatever the exact functional form of the probability distribution is for a particle to move by an amount  $\delta x$  starting at  $x$ , it must satisfy

$$\begin{aligned} \int d(\delta x) P(x, \delta x) &= 1 \\ \int d(\delta x) \delta x P(x, \delta x) &= \langle \delta x \rangle = 0 \\ \int d(\delta x) \delta x^2 P(x, \delta x) &= \langle \delta x^2 \rangle = 2Dt \end{aligned}$$

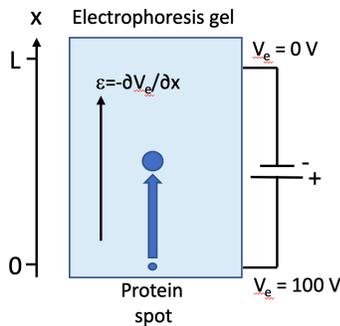
In three dimensions, there are 3 opportunities to move away from the center and

$$\langle \delta r^2 \rangle = 6Dt .$$

**Homework problem S6.1:** A protein has a diffusion coefficient of  $10 \mu\text{m}^2/\text{s}$ . A human cell has a diameter of  $30 \mu\text{m}$ . On average how long does it take a protein to diffuse across a cell?

**In-class exercise:** Assuming a small drug from a dermal patch needs to diffuse through 2 mm of skin to get to the target blood vessels, how long would that take if  $D=100 \mu\text{m}^2/\text{s}$ ? A:  $\sim 5$  h.

**2. Drift velocity** Now let's assume that there is an applied force acting on the particle, in addition to the random force  $F_{\text{random}}$  from the collisions. For example, our protein might be undergoing gel



electrophoresis. It sits in a gel, where it gets bumped randomly by water molecules, but there is also an applied potential or 'voltage'  $V_e = 100$  V in the figure. If the resistance of the gel is constant (see next chapter), the potential difference decreases linearly from  $x = 0$  and  $x = L$  as  $V_e(x) = -x\varepsilon$ , where  $\varepsilon$  is the constant electric field pointing from the positive electrode (anode) to the negative electrode (cathode).

The applied potential causes the energy  $E$  of the charged protein to depend on position:  $E = TS + PV \mu n + V_e Q = TS + \mu n - x\varepsilon Q$ . Note that we need to add a term for the energy of proteins with total charge  $Q$  at electric potential  $V_e(x)$ . The potential pushes positively charged proteins upwards on the gel as shown by the blue arrow.

The position-dependent energy of the protein means its chemical potential also depends on position: remember, molecules go to the lowest chemical potential (highest number of total microstates), so if the protein moves upward, the chemical potential must be lower there. According to our equation,  $\partial E / \partial n = \mu$ , and therefore  $\partial^2 E / \partial n \partial x = \partial \mu / \partial x$ . But also  $\partial E / \partial x = -\varepsilon Q$ , and therefore  $\partial^2 E / \partial x \partial n = -\varepsilon \partial Q / \partial n = -\varepsilon z$ . The derivative  $\partial Q / \partial n = z$  is the molar charge of the protein in units of Coulombs/mole. As you learned in calculus,  $\partial^2 z / \partial x \partial y = \partial^2 z / \partial y \partial x$  (the order of the derivative does not matter), so  $\partial^2 E / \partial n \partial x = \partial^2 E / \partial x \partial n$ , and therefore

$$\frac{\partial \mu}{\partial x} = -\varepsilon z$$

gives us the gradient (position dependence) of the chemical potential as a function of molar charge  $z$  and electric field  $\varepsilon$ . Energy and chemical potential are directly related to the applied force. The applied force is given by  $f = -\partial E / \partial x = \varepsilon Q$ , and similarly, the molar applied force (per mole of protein) is given by  $F_{\text{applied}} = -\partial \mu / \partial x = \varepsilon z$ .

The dragging of the protein in the gel by the electric field is resisted by a friction force  $F_{\text{friction}}$ . The friction is caused by the protein bumping into the gel and water molecules. The faster the protein moves, the more often it bumps and the more friction, so the frictional force increases with velocity  $v$  and opposes the motion of the protein, or  $F_{\text{friction}} = -\gamma v$ .  $\gamma = 1/U$  is the friction coefficient in units of force/velocity or N·s/m, and its inverse ‘ $U$ ’ is known as Einstein’s mobility.

Let’s put those words into an equation, the first step for constructing a model in physical chemistry:

$$\begin{aligned} F &= ma \\ F_{\text{applied}} + F_{\text{friction}} + F_{\text{random}} &= ma \\ -\frac{\partial \mu}{\partial x} - \gamma v + F_{\text{random}} &= ma \approx 0. \end{aligned}$$

If the frictional force is large enough, the inertial acceleration term  $ma$  can be neglected. In that case, the equation simplifies to

$$\frac{\partial \mu}{\partial x} = -\gamma v + F_{\text{random}}.$$

This equation is known as the Langevin equation.

*Case 1.* If the chemical potential  $\mu$  is constant in this equation so that  $\frac{\partial \mu}{\partial x} = 0$  (no applied force), we again just have a particle moving by random Brownian motion, and we get the same answer we already found above:

$$\begin{aligned} \gamma v = F_{\text{random}} &\Rightarrow \frac{\Delta x}{\Delta t} = \frac{F_{\text{random}}}{\gamma} \Rightarrow \frac{\Delta x^2}{\Delta t} = \frac{F_{\text{random}} \Delta x}{\gamma} \Rightarrow \\ \frac{\langle \Delta x^2 \rangle}{\Delta t} = \frac{\langle F_{\text{random}} \Delta x \rangle}{\gamma} &\Rightarrow D \sim \frac{\langle E_{\text{random}} \rangle}{\gamma} \Rightarrow D = \frac{k_B T}{\gamma} \text{ (or } = \frac{RT}{\gamma} \text{ if } \gamma \text{ is in molar units)} \end{aligned}$$

In the first line, we just approximated the derivative by small steps; in the second line, we used the definition of the diffusion coefficient from the previous page, and recall that force·distance = energy and the random thermal energy by equipartition is proportional to  $k_B T$ . Thus,

$$\langle \delta x^2 \rangle = 2 \frac{k_B T}{\gamma} t. \text{ (in 1-D)}$$

The bonus of solving the Langevin equation is that we obtain an expression for the diffusion coefficient  $D = k_B T / \gamma$ : friction and diffusion are not independent of one another. The diffusion coefficient should increase when it gets hotter, and decrease if the friction gets bigger. Those trends make sense intuitively, and it’s always a good thing to check.

*Case 2.* Now let’s do the opposite, and set  $F_{\text{random}} = 0$  and keep the other two terms,

$$\frac{\partial \mu}{\partial x} = -\gamma v = -z\varepsilon,$$

where the last equality is derived on the previous page as the molar force. Then we find that

$$v = \frac{z}{\gamma} \varepsilon = v_{\text{drift}} \text{ if } \gamma \text{ is in molar units.}$$

This is an important equation for gel electrophoresis: it tells us that the protein drifts through the gel at a constant velocity  $v_{\text{drift}}$ . The higher the electric field or charge of the protein, the higher the drift velocity. The higher the friction coefficient, the lower the drift velocity. Bulkier proteins experience higher friction so they drift more slowly and produce a band closer to the origin than smaller proteins. Note that the friction caused by a gel or solution is directly proportional to its viscosity. We won’t prove it here (we could!), but for a spherical particle of radius  $r$ ,  $\gamma = 6\pi r \cdot \eta$ , where  $\eta$  is the viscosity

of the solution. Again, this should be intuitive: the bigger the particle, the larger its friction, and the more viscous the solution, the larger its friction. For water at room temperature,  $\eta = 0.001$  Pa·s (pressure x time units). Note that  $v_{drift}$  does not depend on mass: so the amount the protein moves on the gel is not linearly proportional to mass, although it decreases with protein size.

The ancient Greek philosopher Aristotle believed that particles had to be pushed by a force, or would come to rest. In the 1600s, Newton turned the table on him by showing that  $F_{applied}=ma$ , so acceleration is zero without force, but velocity remains constant. This is true for simple systems of particles that do not encounter any significant friction, like planets moving in the solar system. But Aristotle was kind of right in the end. He guessed that in any real, frictional environment, the Langevin equation would describe motion, not Newton's  $F_{applied}=ma$ . And the Langevin equation does not even have  $ma$  in it! This sometimes happens in science: we come up with a theory, later we think it's wrong, but even later we realize that it was right after all, just in a different context.

**Homework problem 6.2:** A voltage of -100 V is applied to an electrophoresis gel of 10 cm length. The protein phosphoglycerate kinase has charge  $q=+2e$  and in a 1% polyacrylamide gel at room temperature, its diffusion coefficient is  $D=100 \mu\text{m}^2/\text{s}$ . What is the drift velocity of PGK in mm/s? How long should you plan on running the gel if you want the protein to move across half the length of the gel? Let's assume room temperature is 293 K for this problem.

**3. Flux and continuity equation** When particles drift with a velocity  $v$  due to a potential energy  $\mu(x)$  (provides a driving force) and friction (provides a counter force), there is a flux  $J$  of particles, measured in units of moles/s/m<sup>2</sup>. The flux is proportional to the velocity, and to the concentration of particles  $c$  in moles/m<sup>3</sup>, or

$$J = vc$$

But we saw above that  $\frac{\partial\mu}{\partial x} = -\gamma v$  and that  $\gamma=1/U$ , so we can also write this equation as

$$J = -Uc \frac{\partial\mu}{\partial x}.$$

This equation tells us that the flux of particles (such as proteins on the gel) increases with mobility, concentration, and the applied force given by the derivative of the potential.

There is also another important relation between flux and concentration. To derive it, we'll use something from Calc 3 that actually has its uses: Gauss' theorem. The number of particles leaving a volume  $V$  per unit time, given by  $\partial/\partial t \iiint c dV$ , is equal to the flux of particles through the surface around the volume,  $\iint J dA$ , because particle number is conserved, or

$$\frac{\partial}{\partial t} \iiint c(x) dV = \iint \vec{J} \cdot d\vec{A}.$$

Gauss' theorem states

$$\iint \vec{J} \cdot d\vec{A} = \iiint \nabla \cdot \vec{J} dV.$$

Combining the two equations, using the fact that if the integrals are the same, the integrands are the same, and just replacing  $\nabla \cdot \vec{J}$  by  $\partial J/\partial x$  in one dimension, we have.

$$\frac{\partial}{\partial t} c(x, t) = -\frac{\partial}{\partial x} J(x, t) \quad \text{or} \quad \frac{\partial c}{\partial t} + \frac{\partial J}{\partial x} = 0$$

This is the continuity equation. Although it looks pretty abstract, it says something pretty straightforward: if the concentration in a volume is decreasing with time, there must be a position-dependent flux so more molecules flow out of the volume than into the volume. We'll use this when discussing chemical transport and various associated laws in the final part of our tour of pchem.